A Theorem About Uniform Distribution

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Let $\xi_1, \xi_2, \ldots, \xi_n$ be a sequence of independent random variables taking positive integer values and $P\{\xi_j = k\} = \frac{1}{2^k}$, $k \geq 1$. We put $\delta_j = -1$ if k_j is odd, $\delta_j = +1$ if k_j is even, $c_j = c(k_j, \delta_{j-1}) = \frac{2^{k_j}\delta_j - 1 - 3\delta_{j-1}}{6}$ and consider the expression

$$\textstyle\sum_n = \sum(\xi_1, \xi_2, \ldots, \xi_n, \delta_0) =$$

$$= 3^{n-1}c_1 + 3^{n-2}c_2 2^{\xi_1} + 3^{n-3} \cdot c_3\, 2^{\xi_2+\xi_1} +$$

$$+ \ldots + 3c_{n-1} \cdot 2^{\xi_{n-2}+\xi_{n-3}+\ldots+\xi_1} + c_n 2^{\xi_{n-1}+\ldots+\xi_1}$$

Clearly, $\sum_n$ is an integer-valued random variable and for each σ, $0 \leq \sigma < 3^n$, we consider $\sum_n \equiv \sigma (\mathrm{mod}\, 3^{\mathrm{n}})$ and put $\mu_n\left(\frac{\sigma}{3^n}\right) = \sum\limits_{\sum_n \equiv \sigma (\mathrm{mod}\, 3^{\mathrm{n}})} \frac{1}{2^{k_1+k_2+\ldots+k_n+1}}$.

The last summation goes over all values of $\xi_1, \xi_2, \ldots, \xi_n$ and δ_0 which give $\sigma(\mathrm{mod} 3^{\mathrm{n}})$. μ_n is a probability measure on the interval $[0, 1]$. The main result of this paper is the following Theorem.

Main Theorem. *As* $n \longrightarrow \infty$ the *measures* μ_n *converge weakly to the uniform measure.*

The strongest version of this theorem where individual probabilities $\mu_n\left(\frac{\sigma}{3^n}\right)$ converge to $\frac{1}{3^n}$ is wrong. Indeed, one can write down the probability distribution of the first digits in the triadic expansion of $\sum_n$ and see that it is not uniform. A more deep analysis of the distribution of σ_n can be crucial for the progress in the famous $(3x + 1)$-problem.

Proof. The statement of the theorem will follow if we prove that for any integer $\lambda \neq 0$

$$\lim_{n \longrightarrow \infty} \sum_{k_1, \ldots, k_n, \delta_0} \frac{1}{2^{k_1+\ldots+k_n+1}} \exp\left\{2\pi i \frac{\sum_n}{3^n} \lambda\right\} = 0\,. \tag{1}$$

In this expression $k_1, k_2, \ldots, k_n$ are the values of $\xi_1, \xi_2, \ldots, \xi_n$. We may assume that λ is not divisible by 3 and n is even. Other cases require trivial changes.

Denote $k^{(j)} = k_1 + k_2 + \ldots + k_{2j}$, $\ell^{(j)} = k^{(j)} - k^{(j-1)} = k_{2j-1} + k_{2j}$. Fix the values of $\delta_0, \delta_1, \ldots, \delta_n$ and of all $k^{(j)}, j = 1, \ldots, \frac{n}{2}$. With respect to the induced conditional distribution all pairs $(k_{2j-1}, k2j)$ are mutually independent and we can write

$$\sum_{k_1,\ldots,k_n,\delta_0} \frac{1}{2^{k_1+\ldots+k_n+1}} \exp\left\{2\pi i \frac{\sum_n}{3^n}\lambda\right\} =$$

$$= \sum_{\substack{\delta_0,\delta_1,\ldots,\delta_n \\ k^{(1)},k^{(2)},\ldots,k^{(n/2)}}} P\left\{\delta_0,\delta_1,\ldots,\delta_n; k^{(1)},k^{(2)},\ldots,k^{(n/2)}\right\} \prod_{j=1}^{n/2}\varphi_j(\lambda)$$

where $\varphi_j(\lambda)$ is the conditional characteristic function,

$$\varphi_j(\lambda) = \sum_{\text{admissible } k_{2j-1},k_{2j}} \pi\left(k_{2j-1},k_{2j}|\delta_{2j-2},\delta_{2j-1},\ell^{(j)}\right)\cdot$$
$$\cdot \exp\left\{2\pi i \frac{2^{k^{(j-1)}}}{3^{2j-2}}\lambda\left(\frac{c_{2j-1}}{3} + \frac{c_{2j}2^{k_{2j-1}}}{3^2}\right)\right\}, \tag{2}$$

$\pi(k_{2j-1},k_{2j}|\delta_{2j-2},\ \delta_{2j-1},\delta_{2j},\ell^{(j)})$ are the corresponding conditional probabilities. Since $\delta_{2j-1},\delta_{2j}$ are fixed the set of possible pairs (k_{2j-1},k_{2j}) is a subset of the whole set of pairs for which $k_{2j-1}+k_{2j} = k^{(j)} - k^{(j-1)} = \ell^{(j)}$ is given and the conditional distribution is uniform on this subset.

The tables on pages 4 and 5 show these subsets and c_{2j-1},c_{2j} for several first value of $\ell^{(j)}$.

For given $\delta_0,\delta_1,\ldots,\delta_n$ and $k^{(1)},\ldots,k^{(n/2)}$ the index j is called good if $|\varphi_j(\lambda)| \le 1-\frac{1}{n^{\gamma_0}}$ where $\gamma_0 > 0$ is a constant which will be specified later. Otherwise it is called bad. $I^{(g)}$ is the notation for the set of good indices, $I^{(b)} = I \setminus I^{(g)}$.

A sequence $\{\delta_j, 0 \le j \le n\}$, $\{k^{(j)}, 1 \le j \le n\}$ is called good if $|I^{(g)}| \ge n^{\gamma_1}$, where γ_1 is another constant, $\gamma_1 > \gamma_0$. Otherwise, it is called bad.

For good sequences

$$\prod_{j=1}^{n/2}|\varphi_j(\lambda)| \le \prod_{j\in I^{(g)}}|\varphi_j(\lambda)| \le \left(1-\frac{1}{n^{\gamma_0}}\right)^{n^{\gamma_1}} \le \exp\left\{-\,\mathrm{const}\, n^{\gamma_1-\gamma_0}\right\}.$$

The case of bad sequences for which $|I^{(g)}| \le n^{\gamma_1}$ or $|I^{(b)}| \ge n - n^{\gamma_1}$ should be studied in detail.

TABLE 1. $\delta_{2j-2} = -1$.

$\ell^{(j)}$	δ_{2j-1},	δ_{2j}	k_{2j-1},	k_{2j}	c_{2j-1},	c_{2j}	$\frac{c_{2j-1}}{3} + \frac{c_{2j}2^{k_{2j-1}}}{3^2}$
2	-1	-1	1	1	0	0	0
3	-1	1	1	2	0	1	$\frac{1}{9}$
	1	- 1	2	1	1	0	$\frac{1}{3}$
4	- 1	- 1	1	3	0	- 1	$-\frac{2}{9}$
	-1	-1	3	1	- 1	0	$-\frac{1}{3}$
	1	1	2	2	1	0	$\frac{1}{3}$
5	- 1	1	1	4	0	3	$\frac{2}{3}$
	1	- 1	2	3	1	-2	$-\frac{5}{9}$
	- 1	1	3	2	-1	1	$-\frac{5}{9}$
	1	- 1	4	1	3	- 1	$\frac{1}{9}$
6	-1	-1	1	5	0	-5	$-\frac{10}{9}$
	1	1	2	4	0	2	$\frac{8}{9}$
	-1	-1	3	3	-1	-1	$-\frac{11}{9}$
	1	1	4	2	3	0	1
	- 1	- 1	5	1	-5	0	$-\frac{5}{3}$
7	-1	1	1	6	0	11	$\frac{22}{9}$
	1	- 1	2	5	1	-6	-1
	- 1	1	3	4	-1	3	$\frac{7}{3}$
	1	- 1	4	3	3	- 2	$-\frac{23}{9}$
	- 1	1	5	2	- 5	1	$\frac{17}{9}$
	1	- 1	6	1	11	-1	$-\frac{31}{9}$

Table 2. $\delta_{2j-2} = 1$.

$\ell^{(j)}$	δ_{2j-1},	δ_2;	k_{2j-1},	k_{2j}	c_{2j-1},	c_{2j}	$\frac{c_{2j-1}}{3} + \frac{c_{2j}-2^{k_{2j-1}}}{3^2}$
2	-1	-1	1	1	-1	0	$\frac{1}{3}$
3	-1	1	1	2	-1	1	$-\frac{1}{9}$
	1	-1	2	1	0	-1	$-\frac{4}{9}$
4	- 1	-1	1	3	-1	-1	$-\frac{5}{9}$
	-1	-1	3	1	-2	0	$\frac{1}{3}$
	1	1	2	2	0	0	0
5	- 1	1	1	4	-1	3	$-\frac{1}{9}$
	1	-1	2	3	0	-2	$-\frac{8}{9}$
	-1	1	3	2	-2	1	$\frac{2}{9}$
	1	-1	4	1	2	-1	$\frac{10}{9}$
6	-1	-1	1	5	-1	-5	$-\frac{13}{9}$
	1	1	2	4	0	2	$\frac{8}{9}$
	-1	-1	3	3	-2	-2	-3
	1	1	4	2	2	0	$\frac{2}{3}$
	-1	-1	5	1	-6	0	-2
7	-1	1	1	6	- 1	11	$\frac{19}{9}$
	1	- 1	2	5	0	-6	$-\frac{24}{9}$
	-1	1	3	4	-2	3	2
	1	-1	4	3	2	-2	$-\frac{26}{g}$
	-1	1	5	2	-6	1	$\frac{46}{9}$
	1	-1	6	1	10	-1	$-\frac{34}{9}$.

We have

$$\varphi_j(\lambda = \sum_{\text{admissible } (k_{2j-1}, k_{2j})} \exp\left\{\frac{2\pi i\, 2^{k^{(j-1)}}\lambda}{3^{2j-2}}\left(\frac{c_{2j-1}}{3} + \frac{c_{2j}\, 2^{k_{2j-1}}}{3^2}\right)\right\}\cdot$$

$$\cdot\pi\left(k_{2j-1}, k_{2j} \mid \delta_{2j-2},\, \delta_{2j-1},\, \delta_{2j},\, \ell^{(j)}\right) =$$

$$= 1 - \sum_{\text{admissible } (k_{2j-1}, k_{2j})} \left(1 - \exp\left\{2\pi i\, \tfrac{2^{k^{(j-1)}}\lambda}{3^{2j-2}}\left(\tfrac{c_{2j-1}}{3} + \tfrac{c_{2j}2^{k_{2j-1}}}{3^2}\right)\right\}\right)\cdot$$

$$\cdot\pi\left(k_{2j-1}, k_{2j} \mid \delta_{2j-2},\, \delta_{2j-1},\, \delta_{2j}, \ell^{(j)}\right).$$

All expressions $1 - \exp\left\{2\pi i\, \tfrac{2^{k^{(j-1)}}\lambda}{3^{2j-2}}\left(\tfrac{c_{2j-1}}{3} + \tfrac{c_{2j}2^{k_{2j-1}}}{3^2}\right)\right\}$

have positive real parts. Therefore, if j is bad and $2 \le \ell^{(j)} \le 7$ then there should be

$$\left|\exp\left\{2\pi i\, \frac{2^{k^{(j-1)}}\lambda}{3^{2j-2}}\left(\frac{c_{2j-1}}{3} + \frac{c_{2j}2^{k_{2j-1}}}{3^2}\right)\right\} - 1\right| \le \frac{\text{const}}{n^{\gamma_0}} \tag{3}$$

Here and further const is an absolute constant whose exact value plays no role in the proof.

Let us write

$$\frac{2^{k^{(j-1)}}\lambda}{3^{2j-2}} = 3^s\, m_2^{(j-1)} + 3\, m_1^{(j-1)} + m_0^{(j-1)} + \theta_j$$

where $s \ge 2$, $m_0^{(j-1)}$ and $m_1^{(j-1)}$ are integers, $0 \le m_0^{(j-1)},\ m_1^{(j-1)} \le 2$, $m_2^{(j-1)}$ is not divisible by 3 and $|\theta_j| \le \frac{1}{2}$.

Denote by $A_1(\{\delta_j, 0 \le j \le n\},\ \{k^{(j)}, 1 \le j \le n\})$ the set of indices j for which $\delta_{2j-2} = -1$, $\delta_{2j-1} = 1$ $\delta_{2j} = -1$, $\ell^{(j)} = 5$ or $\delta_{2j-2} = 1$, $\delta_{2j-1} = -1$, $\delta_{2j} = 1$, $\ell^{(j)} = 5$. In both cases one term in the expression for $\varphi_j(\lambda)$ has k_{2j-1}, k_{2j} with $\frac{c_{2j-1}}{3} + \frac{c_{2j}2^{k_{2j-1}}}{3^2} = \pm\frac{1}{9}$ (see Tables 1 and 2). Therefore,

$$\left(3m_1^{(j-1)} + m_0^{(j-1)} + \theta_j\right)\left(\frac{c_{2j-1}}{3} + \frac{c_{2j}2^{k_{2j-1}}}{3^2}\right) = \pm\left(3m_1^{(j-1)} + m_0^{(j-1)} + \theta_j\right)\frac{1}{9}$$

and in order that (3) were valid for this term we should have $m_0^{(j-1)} = m_1^{(j-1)} = 0$, $|\theta_j| \le \frac{\text{const}}{n^{\gamma_0}}$.

Denote by B_1 the set of sequences $\{\delta_j, 0 \le j \le n\}$, $\{k^{(j)}, 1 \le j \le n\}$ for which $|A_1| \ge b_1 n$. The probability of the complement to B_1 is exponentially small if b_1 is sufficiently small.

Since we consider bad sequences the majority of $j \in A_1(\{\delta_j, 0 \le j \le n\}, \{k^{(j)}, 1 \le j \le \frac{n}{2}\})$ consists of bad j. For bad $j \in A_1(\{\delta_j, 0 \le j \le n\}, \{k^{(j)}, 1 \le j \le \frac{n}{2}\})$ we have the representation (3) with $m_0 = m_1 = 0$ and $|\theta_j| \le \frac{\text{const}}{n^{\gamma_0}}$.

Assume that

$$\frac{2^{k^{(j-1)}}\lambda}{3^{2j-2}} = 3^s\, m_2^{(j-1)} + \theta_j \tag{4}$$

for some integer $m_2^{(j-1)} \ge 1$ not divisible by 3, $s \ge 2$, $|\theta_j| \le \frac{\text{const}}{n^{\gamma_0}}$. Then

$$\frac{2^{k^{(j)}}\lambda}{3^{2j}} = 3^{s-2} \cdot 2^{\ell^{(j)}}\, m_2^{(j-1)} + \frac{2^{\ell^{(j)}}}{3^2}\,\theta_j\,. \tag{5}$$

Thus (5) gives the same representation as (4) for $\frac{2^{k^{(j)}}\lambda}{3^{2j}}$ with $s' = s-2$, $m_2^{(j)} = 2^{\ell^{(j)}} \cdot m_2^{(j-1)}, \theta_{j+1} = \frac{2^{\ell^{(j)}}}{3^2}\,\theta_j$.

A sequence of indices j, $j_1 \le j \le j_2$, is called a cycle if

i) for all j, $j_1 \le j \le j_2$, the representation

$$\frac{2^{k^{(j)}}\lambda}{3^{2j}} = 3^{s_j}\, m_2^{(j)} + \theta_j$$

with $s_j \ge 2$, $|\theta_j| \le b_2$ is valid where b_2 is another sufficiently small constant (see below);

ii) for $j = j_1 - 1$ and $j = j_2 + 1$ it is not valid and $\delta_{2j_2}, \delta_{2j_2+1}, \delta_{2j_2+2}, \ell^{(j_2+1)}$ are such that $\ell^{(j_2+1)} \le 7$ and at least one term in (2) is such that $\frac{c_{2j_2}+1}{3} + \frac{c_{2j_2}2^{k_{2j_2}+1}}{3^2} = \frac{t}{3^2}$ where t is an integer not divisible by 3 (see Tables 1 and 2).

Lemma 1. *There exists a constant $\alpha, 0 < \alpha < 1$, such that for any cycle $[j_1, j_2]$*

$$|\varphi_{j_2+1}(\lambda)| \le 1 - \alpha\,.$$

Proof. A point j_2 can be the right end of a cycle by one of the following two reasons.

1. For $j = j_2$

$$\frac{2^{k^{(j_2)}}\lambda}{3^{2j_2}} = 3^2\, m_2^{(j_2)} + \theta_{j_2} \text{ or } \frac{2^{k^{(j_2)}}}{3^{2j_2}} = 3^3 \cdot m_2^{(j)} + \theta_{j_2}$$

with $|\theta_{j_2}| \le b_2$. Then

$$\frac{2^{k^{(j_2+1)}}\lambda}{3^{2(j_2+1)}} = 2^{\ell^{(j_2+1)}}\, m_2^{(j_2)} + \theta_{j_2}\, \frac{2^{\ell^{(j_2+1)}}}{3^2} \text{ or } \frac{2^{k^{(j_2+1)}}\lambda}{3^{2(j_2+1)}} = 3\cdot 2^{\ell^{(j_2+1)}}\, m_2^{(j_2)} + \theta_{j_2}\, \frac{2^{\ell^{(j_2+1)}}}{3^2}\,.$$

Since $\ell^{(j_2+1)} \le 7$ we have $|\theta_{j_2}\, \frac{2^{\ell^{(j_2+1)}}}{3^2}| \le \text{const}\, b_2$. Any product $2^{\ell^{(j_2+1)}} \cdot m_2^{(j_2)}$ or $3\cdot 2^{\ell^{(j_2+1)}}\, m_2^{(j_2)}$ is not divisible by 9. In view of ii) both products

$$2^{\ell^{(j_2+1)}}\, m_2^{(j_2)} \left(\frac{c_{2j+1}}{3} + \frac{c_{2j+2}2^{k_{2j+1}}}{3^2}\right) \text{ or } 3\cdot 2^{\ell^{(j_2+1)}} m_2^{(j_2)} \left(\frac{c_{2j+1}}{3} + \frac{c_{2j+1}2^{k_{2j+1}}}{3^2}\right)$$

are fractions with the denominator 3 or 9. If b_2 is small enough then

$$\left|1 - \exp\left\{2\pi i\, \frac{2^{k^{(j_2+1)}}\lambda}{3^{2(j_2+1)}} \left(\frac{c_{2j+1}}{3} + \frac{c_{2j+2}2^{k_{2j+1}}}{3^2}\right)\right\}\right| \ge \alpha_1$$

for some constant $\alpha_1 > 0$. This gives the statement of the lemma in this case.

2. For $j = j_2$

$$\frac{2^{k^{(j_2)}}\lambda}{3^{2j_2}} = 3^s\, m_2^{(j_2)} + \theta_{j_2}$$

where $s \ge 3, |\theta_{j_2}| \le b_2$ and

$$\frac{2^{k^{(j_2+1)}}\lambda}{3^{2(j_2+1)}} = 3^{s-2} \cdot 2^{\ell^{(j_2+1)}}\, m_2^{(j_2)} + \theta_{j_2} \cdot \frac{2^{\ell^{(j_2+1)}}}{3^2}$$

with $\left|\theta_{j_2} \cdot \frac{2^{\ell^{(j_2+1)}}}{3^2}\right| \ge b_2$. Since $\ell^{(j_2+1)} \le 7$ we have $\left|\theta_{j_2} \cdot \frac{2^{\ell^{(j_2+1)}}}{3^2}\right| \le b_2\,\text{const}$. In this case

$$\left|\exp\left\{2\pi i\, \frac{2^{k^{(j_2+1)}}\lambda}{3^{2(j_2+1)}} \left(\frac{c_{2j+1}}{3} + \frac{c_{2j+2}}{3^2} \cdot 2^{k_{2j+1}}\right)\right\} - 1\right|$$

$$= \left|\exp\left\{2\pi i\,\theta_{j_2} \cdot \frac{2^{\ell^{(j_2+1)}}}{3^2} \cdot \left(\frac{c_{2j+1}}{3} + \frac{c_{2j+2}}{3^2}\, 2^{k_{2j+1}}\right)\right\} - 1\right| \ge \alpha_2$$

for another constant $\alpha_2 > 0$. Lemma is proven.

We shall prove that with probability tending to 1 as $n \longrightarrow \infty$ the number of cycles is not less than $\alpha_3 \, \ell n \, n$ for another constant α_3. In view of Lemma 1, this gives the estimate $\left|\prod_{j=1}^{n/2} \varphi_j(\lambda)\right| \leq (1-\alpha)^{\alpha_3 \, \ell n \, n} = \frac{1}{n^{\gamma_2}}$ with $\gamma_2 = -\alpha_3 \, \ell n(1-\alpha)$.

A segment $[j_1, j_2]$ is called pre-cycle if for all j, $j_1 \leq j \leq j_2$

i$'$) the representation

$$\frac{2^{k^{(j)}}\lambda}{3^{2j}} \, 3^{s_j} \, m_2^{(j)} + \theta_j$$

with $s_j \geq 2$ and $m_2^{(j)}$ not divisible by 3, $|\theta_j| \leq b_2$ is valid.

ii$''$) for $j = j_1 - 1$, $j = j_2 + 1$ it is not valid. Any point j with the property i$'$) can be included in a unique way in a pre-cycle. The difference $j_2 - j_1 = d([j_1, j_2])$ is called the length of the pre-cycle. It is clear that $\theta_j = \frac{2^{k^{(j)} - k^{(j_1)}}}{3^{2(j-j_1)}} \theta_{j_1}$ for $j \in [j_1, j_2]$ and $|\theta_{j_1}| \geq \frac{1}{3^{2j_1}}$. Therefore the following lemma holds.

Lemma 2. *There exist positive constants α_4, α_5 such that for given j_1, the conditional probability that $d([j_1, j_2]) \geq \alpha_4 \, j_1$ is less than* $\exp\{-\alpha_5 \, j_1\}$.

Proof. Assuming that α_4 is chosen consider the situation $d([j_1, j_2]) \geq \alpha_4 \, j_1$. Then for $j - j_1 = [\alpha_4 \, j_2]$

$$\frac{2^{k^{(j)} - k^{(j_1)}}}{3^{2(j-j_1)}} \cdot |\theta_{j_1}| \leq b_2$$

which implies

$$2^{k^{(j)} - k^{(j_1)}} \leq \frac{b_2 \cdot 3^{2(j-j_1)}}{|\theta_{j_1}|} \leq b_2 \cdot 3^{2j}$$

since $|\theta_{j_1}| \geq \frac{1}{3^{2j_1}}$. This gives

$$k^{(j)} - k^{(j_1)} \leqq \frac{\ell n \, b_2 + 2j \, \ell n \, 3}{\ell n \, 2} = \frac{\ell n \, b_2 + 2j_1 \, \ell n \, 3 + 2\alpha_4 \, j_1 \, \ell n \, 3 + O(1)}{\ell n \, 2} = \tag{6}$$
$$= \frac{j_1 (2 \, \ell n \, 3 + 2\alpha_4 \, \ell n \, 3) + \ell n \, b_2 + O(1)}{\ell n \, 2}$$

In a typical situation $k^{(j)} - k^{j_1)}$ grows as $2(j - j_1)$ which is equivalent to $4\alpha_4 j_1$ while the main term in the last expression grows as $\frac{2\ell n \, 3 + 2\alpha_4 \ell n \, 3}{\ell n \, 2} \cdot j_1$. Therefore, for large enough α_4 we have the inequality $\frac{2\ell n \, 3 + 2 \, \alpha_4 \, \ell n \, 3}{\ell n \, 2} < 4 \, \alpha_4$ and the probability of the sequences

$k^{(j)} - k^{(j_1)}$ satisfying (6) can be estimated with the help of the usual methods in the theory of probabilities of large deviations. Lemma is proven.

Consider the segment $[n^{\gamma_3}, n]$ for any $\gamma_3, 0 < \gamma_3 < 1$. The value of γ_3 will determine the estimate of some probabilities below. It follows easily from Lemma 2 and from the fact that the majority of the indices j is bad that with probability tending to 1 the number of pre-cycles which intersect $[n^{\gamma_3}, n]$ is greater than $\alpha_6 \, \ell n \, n$ for another constant $\alpha_6 > 0$.

The difference between pre-cycles and cycles is in the behavior at the right end-point. Suppose that j_1 is the beginning of a pre-cycle, $m_2(j_1)$, θ_{j_1} are the corresponding parameters of the initial point. Under this condition the conditional probability that a pre-cycle is a cycle is greater than some constant $\alpha_7 > 0$. By this reason with probability tending to 1 the number of cycles is greater than $\alpha_8 \, \ell n \, n$ for some constant $\alpha_8 > 0$. This implies (1). Theorem is proven.

The proof presented in this paper is an improvement of the proof of a similar statement given in [1]. It gives the power-like decay of the conditional characteristic function. However, presumably its actual decay is exponential.

The same methods allow to prove the main theorem for conditional distributions of $\xi_1 \ldots \xi_n$ under conditions $\xi_1 + \xi_2 + \ldots + \xi_n = k$, $|k - 2n| = O(\sqrt{n})$.

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Reference

1. Ya. G. Sinai, "Uniform Distribution in the $(3x+1)$-Problem," *Moscow Mathematical Journal*, in press.