Mathematical Hydrodynamics

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§1. Introduction

Mathematical hydrodynamics deals basically with Navier-Stokes and Euler systems. In the d-dimensional case and incompressible fluids these are the systems of $(d+1)$ equations for d-dimensional velocity vector $u = (u_1, \dots, u_d)$ and a scalar function p called the pressure. In the Navier-Stokes (NSS) case it has the form:

$$\frac{\mathcal{D}u}{dt} = \nu \Delta u - \nabla p + f(x,t), \tag{1}$$

$$\operatorname{div} u = 0 \tag{2}$$

where $\nu > 0$ is the viscosity, $\frac{\mathcal{D}u}{dt}$ is the vector with the components $\frac{\partial u_i}{\partial t} + \sum_{k=1}^{d} \frac{\partial u_i}{\partial x_k} u_k$ and $f(x,t)$ is the vector of external forces which is assumed to be a given function of x and t. Depending on the boundary conditions the system (1), (2) can be considered on the whole space R^d, on the d-dimensional torus Tor^d or on a compact domain O. In the latter case one usually imposes "sticking" boundary condition $u|_{\partial O} = 0$.

If $\nu = 0$ the system (1), (2) becomes Euler system of equations which describes the dynamics of the ideal liquid. In this paper we shall consider only NSS (1) and (2) with $\nu > 0$. The Euler system has a very rich structure but we shall not discuss its properties here. It deserves a special consideration.

NSS is quite rigid. The only natural parameters are the viscosity $\nu > 0$ and the external forces $f(x,t)$ which may be random. By this reason the analysis of (1), (2) is usually very difficult. The problem of turbulence can be formulated as a problem of asymptotic regimes of solutions of NSS as $\nu \to 0$. We believe that some of the methods and results which we shall discuss below can be useful for this notoriously difficulty problem as well.

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Mathematical hydrodynamics consists of three different parts:

1. One-dimensional hydrodynamics.
2. Two-dimensional hydrodynamics.
3. Three or more-dimensional hydrodynamics.

Each part has its own methods and special features.

We shall discuss each part separately.

§2. One-dimensional Hydrodynamics

Under one-dimensional hydrodynamics we understand the theory of Burgers system of equations. It is written for d-dimensional unknown vector $u = (u_1, \ldots, u_d)$

$$\frac{\mathcal{D}u}{dt} = \nu \Delta u + f(x,t) \tag{3}$$

and differs from (3) by the absence of the pressure term which changes the whole structure significantly. Remark that the Burgers system can be written for any d but is mostly developed for $d = 1$. For any d the system (3) has an invariant sub-manifold of gradient-like solutions on which it can be reduced to a heat equation with the help of the so-called Hopf-Cole substitution. Outside this manifold, solutions of the Burgers system are as complicated as solutions of the whole NSS.

We shall consider gradient-like solutions of (3) with periodic boundary conditions and random forcing. Since we are dealing with gradient-like solutions we assume that the forcing is also the gradient, i.e. $f(x,t) = \nabla F(x,t)$ where F is the potential. The main case here is $F(x,t) = \sum_{|k|\leq K} B_k(t) \cdot \exp\{2\pi i < k, x >\}$ which means that the force acts on finitely many modes. In the random case $B_k(t)$ are independent "white noises" except some relation which ensures that F is real-valued.

(3) can be considered as a stochastic PDE describing a Markov process with continuous time in the space of continuous in time, periodic in x functions. It was proven in [Si1] that this Markov process has a unique stationary measure and the corresponding dynamical system is ergodic and has strong properties of mixing.

Much more interesting case appears when $\nu = 0$ which is actually the limit $\nu \downarrow 0$. For $d = 1$, we are dealing with a single quasi-linear equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial (u^2)}{\partial x} = \frac{\partial}{\partial x}\left[\sum B_k(t)\, e^{2\pi i k x}\right] \tag{4}$$

and periodic boundary condition of period 1. Typical solutions of (4) have discontinuities which are shock waves. Therefore, a stationary measure for (4) if it exists, should be a probability measure on the Borel σ-algebra of the Skorokhod space of functions with discontinuities of the first kind.

This measure was constructed in the paper [EKMS]. We used the so-called Lax-Oleinik variational method to implement the so-called "One Force - One Solution" principle. It means that for a typical realization of the random force we find a particular solution of (4) which by this reason is also random and is a global attractor in the sense that an arbitrary solution converges to it.

We now give the description of this solution (details can be found in [EKMS]). Take an arbitrary piece-wise differentiable function $X = \{x(t), -\infty < t \leq 0\}$, with values in S^1 and consider the formal "action"

$$A\{X\} = \int_{-\infty}^{s} \left[\frac{1}{2}\,\dot{x}^2(t) + F(x(t), t)\right] dt$$

where F is the potential of external forcing. We call $\bar{X}$ a one-sided minimizer if A increases if we replace $\bar{X}$ by an arbitrary perturbation on any compact interval, preserving the initial point. One of the first results of [EKMS] says that with probability 1 for every $y \in S^1$ there exists a one-sided minimizer $\bar{X}_y$ for which $\bar{X}(0) = y$. Then the attractive solution has the form $u(x,s) = \frac{d\bar{X}(s)}{dt}$, $\bar{X}(s) = x$. It has many remarkable properties which surprisingly are proven with the help of hyperbolic theory of dynamical systems. For example, one can show that with probability 1 the number of points where $u(x,s)$ is discontinuous is finite. However, it is not known whether the expectation of this number is finite.

Presumably, statistical properties of solutions of the Burgers equation on the whole line are more subtle. K. Khanin and Viet.Ha [KV] could extend the theory of [EKMS] to some class of potentials $F(x,t)$ on the whole line. In the paper by Itturiaga and Khanin [IK] one could find extensions of the one-dimensional theory to the multi-dimensional one. It is worth mentioning that the case of $F(x,t)$ being white noise in space and time is out of reach. For this case Bouchaud, Mezard and Parisi [BMP] suggested that stationary measure is white noise in space.

§3. Two-dimensional NSS

The incompressibility condition $\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0$ allows us to write down the equation for vorticity $\omega(x,t) = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$ instead of NSS. In the case of periodic boundary conditions it is convenient to use Fourier series. Then the infinite-dimensional system of equations for the evolution of Fourier modes $\hat{\omega}_k(t) k \in \mathbb{Z}^2$ takes the form

$$\frac{\partial \hat{\omega}_k(t)}{\partial t} = -\nu |k|^2 \hat{\omega}_k(t) - 2\pi i \sum_{\ell} \hat{\omega}_\ell(t)\, \hat{\omega}_{k-\ell}(t)\, \frac{\langle k, \ell^\perp \rangle}{\langle \ell, \ell \rangle} + \hat{f}_k(t)\,. \tag{5}$$

where $\hat{f}_k(t)$ are the Fourier coefficients of the vorticity of the external force.

In the two-dimensional case and periodic boundary conditions the so-called energy and enstrophy inequalities ensure the fast decay (in k) of $\hat{\omega}_k(t)$ at infinity. The first result of this type was proven by C. Foias and R. Temam in [FT]. In our joint paper with J. Mattingly [MS] we proposed another proof of a similar result (see also the monograph by G. Gallavotti [G]). Here is the formulation (see [ES]).

Theorem 1. *Let $|\hat{\omega}_k(0)| \le \frac{\mathcal{D}}{|k|^\gamma}$ for some $\gamma > 1$ and finite $\mathcal{D}$ and (for simplicity) $\hat{f}_k(t)$ is different from zero only for finitely many k and bounded. Then for some constant $a > 0, t_0 > 0$, another constant $\mathcal{D}'$ and all $0 \le t \le t_0$*

$$|\hat{\omega}_k(t)| \le \frac{\mathcal{D}'}{|k|^\gamma} e^{-a\cdot t\cdot |k|}\,.$$

For $t > t_0$

$$|\hat{\omega}_k(t)| \le \frac{\mathcal{D}'}{|k|^\gamma} e^{-a_1\cdot |k|}$$

for some constant $a_1 > 0$.

Surprisingly enough, this theorem cannot be applied to the NSS on the whole plane. Here we are dealing with the non-linear integral equation

$$\frac{\partial \hat{\omega}(k,t)}{\partial t} = -\nu |k|^2 \hat{\omega}(k,t) + \frac{1}{2\pi} \int \hat{\omega}(\ell,t)\, \hat{\omega}(k-\ell,t)\, \frac{\langle k, \ell^\perp \rangle}{|\ell|^2}\, d\ell + \hat{f}(k,t)\,. \tag{6}$$

In the recent paper by Arnold, Bakhtin and Dinaburg [ABD] it was shown that under some assumptions on initial conditions solutions of (6) satisfy the inequalities

$$|\hat{\omega}(k,t)| \le \mathcal{D} \left(\frac{1}{|k|^\gamma} + 1 \right) e^{-a(t)|k|} \tag{7}$$

for some constants $\mathcal{D} < \infty$, $\gamma > 0$ and $a(t) = a \cdot t$ for $0 \le t \le t_0$, $a(t) = a_1 =$ const for $t > t_0$.

The estimate (7) shows that solutions can grow near $k = 0$ and decay exponentially outside a neighborhood of $k = 0$.

The growth of solution near $k = 0$ is related to the flow of energy toward small k which is called the energy cascade. (See e.g.[DG]). However, the asymptotics of the size of the neighborhood where this happens remains an open question.

§4. Three-dimensional NSS

For 3D-NSS the existence and uniqueness results for classical solutions were proven for sufficiently small or large initial conditions (see the recent survey paper by M. Cannone [C] and the issue of *Russian Math. Surveys*, vol. 58, N2 dedicated to O. Ladyzenskaya.

In our joint paper with Dinaburg [DS1] we proposed some modification of the NSS on the whole space which shows similar process of non-linear spreading as NSS but looks like a quasi-linear system with non-local coefficients. The system in Fourier space is defined for functions $v(k) \in R^3$, $k \in R^3$, $v(k) \perp k$ and has the form:

$$\begin{aligned} \frac{dk}{dt} &= A\,k\,, \\ \frac{dv}{dt} &= -\nu|k|^2\,v \,+\, A^*v \,-\, \Pi_k\,A^*\,v \end{aligned} \tag{8}$$

where Π_k is the reflection off the plane orthogonal to k, i.e. $\Pi_k\,e \,=\, e - 2\left(\frac{k}{|k|}\,,\,e\right)\frac{k}{|k|}$.

This system appeared earlier in the works of physicists A. Obukhov, F. Dolzanskii et. al. but they did not consider this system in connection with the problem of existence and uniqueness of solutions.

Recently in our joint paper with Dinaburg, we could investigate the character of decay of solutions of 3-NSS on the whole space when the initial conditions have some singularities near $k = 0$ and $k = \infty$. This will be discussed in a separate paper.

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