On some approximation of the 3$\mathcal{D}$-Euler System

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Dedicated to the memory of M. Herman

In our joint paper [1] we proposed a quasi-linear approximation for the 3$\mathcal{D}$-Navier-Stokes system on the whole space R^3. It was constructed for Fourier transforms $\tilde{u}(k,t) = \{\tilde{u}_1(k,t)\,,\tilde{u}_2(k,t)\,,\tilde{u}_3(k,t)\}$ of the incompressible vector fields $u(x,t) = (u_1(x,t),$ $u_2(x,t),\ u_3(x,t)))$, div $u = 0$. Here $x = (x_1, x_2, x_3) \in R^3$, $k = (k_1, k_2, k_3) \in R^3$ and $\tilde{u}$ is pure imaginary and odd. Putting $\tilde{u}(k,t) = i\,v(k,t)$ with $v(-k,t) = -v(-k,t)$ we can write our approximation as a system of quasi-linear equations (see [1]):

$$\frac{\partial v_i(k,t)}{\partial t} + \sum_{s=1}^{3} B_s \frac{\partial v_i(k,t)}{\partial k_s} = A_i^{(\nu)}\,, i = 1,2,3 \tag{1}$$

$$\sum_{i=1}^{3} k_i\, v_i(k,t) = 0 \tag{2}$$

where

$$A_i^{\nu} = A_i^{(\nu)}\,(k,v,t) = -\nu|k|^2\, v_i(k,t) +$$
$$+\sum_{j=1}^{3} v_j(k,t)\, a_{ji}(t) - \frac{2\,k_i}{|k|^2}\sum_{j=1}^{3}\sum_{\ell=1}^{3} v_j(k,t)\, k_\ell\, a_{j\ell}(t)\,,$$

and $a_{ij}(t) = \int_{R^3} k_i\, v_j(k,t) dk$ are the first moments of v_j with respect to the space variables, $\nu > 0$ is the viscosity and $B_s = B_s(k,t) = \sum_{j=1}^{3} a_{sj}(t) k_j$.

By putting $\nu = 0$ we get a similar approximation for the Euler system describing the dynamics of the free ideal liquid in R^3.

In what follows we consider the system (1) and (2) for $\nu \geq 0$ satisfying at $t = 0$ the incompressibility condition (2). As was shown in [1] any solution $v(k,t)$ satisfies the incompressibility condition (2) for all t for which a solution exists.

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The system (1) has characteristics which are described by the system of equations

$$\frac{d\,k_i(t)}{dt} = B_i\,, \quad i = 1,2,3 \tag{3}$$

Along solutions of (3) the system (1) becomes a system of ordinary differential equations

$$\begin{aligned}\frac{dv_i(t)}{dt} &= -\nu|k|^2\, v_i(t) + \sum_{j=1}^{3} v_j(t)a_{ji}(t) - \\ &- \frac{2k_i}{|k|^2} \sum_{j=1}^{3}\sum_{\ell=1}^{3} v_j(t)k_\ell \cdot a_{j\ell}(t),\ i = 1,2,3\end{aligned} \tag{4}$$

$$\sum_{j=1}^{3} k_j(t)\, v_j(t) = 0\,.$$

The system (3) and (4) has finite-dimensional versions if we assume that initial conditions $v(k,0)$ are non-zero only for a finite set of points $k^{(n)} = k^{(n)}(0)$, $1 \le n \le N$, and the integrals giving $a_{ij}(t)$ are replaced by the Riemannian sums $A_{ij}(t) = \frac{1}{N}\sum_{n=1}^{N} k_i^{(n)}(t)v_j^{(n)}(t)$.

Consider the 3×3-matrix $A(t) = \{a_{ij}(t)\}$. Then our finite-dimensional system of $\mathcal{ODE}$ which is a system of $6\,N$ equations can be written in the form:

$$\frac{dk^{(m)}(t)}{dt} = A(t)\,k^{(m)}(t) \tag{5}$$

$$\begin{aligned}\frac{dv^{(m)}(t)}{dt} &= -\nu\,|k^{(m)}(t)|^2\, v^{(m)}(t) + A^t(t)v^{(m)}(t) \\ &- \frac{2\,k^{(m)}(t)}{|k^{(m)}(t)|^2}\,(v^{(m)}(t),\, A(t)\,k^{(m)}(t))\end{aligned} \tag{6}$$

$$(k^{(m)}(t),\, v^{(m)}(t)) = 0,\ 1 \le n \le N\,,$$

where A^t is the transposed of A.

Some properties of the system (5) and (6).

1) Matrix elements of $A(t)$ do not depend explicitly on ν.

2) $tr\, A(t) \,=\, 0$.

3) For any three numbers $1 \le p, q,\; r \;\le N$ the oriented volume $\mathcal{D}(p,q,r)$ of the parallel piped generated by $k^{(p)}(t)$, $k^{(q)}(t)$, $k^{(r)}(t)$ is the first integral of (5) and (6).

4) $\det A(t)$ depends only on $v^{(m)}(t)$, $1 \le m \le N$.

The property 1) is obvious, the properties 2) and 3) are proven in [1]. To prove the property 4) we remark that

$$\det A(t) \,=\, \frac{1}{N^3} \sum_{1\le \ell, m, n \le N} \; \sum_{(v_{i_1}, v_{i_2}, v_{i_3})} k_1^{(\ell)}\, v_{i_1}^{(\ell)}\, k_2^{(m)} \,\cdot\, v_{\ell_2}^{(m)} \,\cdot$$

$$\cdot\, k_3^{(n)}\, {v_{i_3}}^{(n)}\, (-1)^{\sigma(i_1,i_2,i_3)} \;=\; \frac{1}{N^3} \sum_{1\le \ell, m, n\le N} \mathcal{D}(\ell, m, n)\, v_1^{(\ell)}\, v_2^{(m)}\, v_3^{(n)}$$

where the second sum is taken over all permutations of $(1,2,3)$ and $\sigma(i_1,i_2,i_3) \,=\, 0$ for even permutations and 1 for odd. Thus the property 4 is proven.

In [2] the case of two particles $N = 2$ was studied for $\nu > 0$. The main result of [2] was the following theorem.

Theorem 1: *Let* $\nu \,>\, 0$. *For any initial condition* $k^{(1)}(0) \,\neq\, 0$, $k^{(2)}(0) \,\neq\, 0$, $v^{(1)}(0)$, $v^{(2)}(0)$ *satisfying (6) the solution of (5) exists for all* $t > 0$ *and*

1) there exist non-zero vectors $k^{(1)}(\infty)$, $k^{(2)}(\infty)$, *depending on initial conditions such that*
$\lim\limits_{t\longrightarrow\infty} k^{(i)}(t) \,=\, k^{(i)}(\infty)$, $i = 1, 2\,;$

2) $\lim\limits_{t\longrightarrow\infty} v^{(i)}(t) \,=\, 0\,, i = 1, 2.$

Beginning with this moment we consider $N = 2$ and $\nu = 0$. The system (5) can be written in a more compact form:

$$
\begin{aligned}
\frac{d\,k_i^{(1)}(t)}{dt} &= k_i^{(2)}(t)\,V^{(2)}(t)\,, \\
\frac{d\,k_i^{(2)}(t)}{dt} &= k_i^{(1)}(t)\,V^{(1)}(t)\,, \\
\frac{d\,v_i^{(1)}(t)}{dt} &= v_i^{(2)}(t)\,V^{(1)}(t) - \frac{4\,k_i^{(1)}(t)}{|k^{(1)}(t)|^2}\,V^{(1)}(t)\,V^{(2)}(t) \\
\frac{d\,v_i^{(2)}(t)}{dt} &= v_i^{(1)}(t)\,V^{(2)}(t) - \frac{4\,k_i^{(2)}(t)}{|k^{(2)}(t)|^2}\,V^{(1)}(t)\,V^{(2)}(t)\,,
\end{aligned}
\tag{7}
$$

$i = 1, 2, 3$

where $V^{(1)}(t) = (k^{(2)}(t)\,, v^{(2)}(t))$, $V^{(2)}(t) = (k^{(1)}(t)\,, v^{(2)}(t))$ and incompressibility condition

$$\sum_{i=1}^{3} k_i^{(j)}(t)\,v_i^{(j)}(t) = 0\,, j = 1, 2\,. \tag{8}$$

Lemma 1: *The components of the vector* $J = (J_1, J_2, J_3) = [k^{(1)}(t), k^{(2)}(t)]$ *and* $I_1 = |k^{(1)}(t)|^2\,V^{(1)}(t)$, $I_2 = |k^{(2)}(t)|^2\,V^{(2)}(t)$ *are the first integrals of the system (7) and (8).*

The first three integrals are also the first integrals for $\nu > 0$ and the proof remains the same (see [2]). I_1 and I_2 are the first integrals only for $\nu = 0$. The statement follows from direct checking.

Lemma 2: *There exists a function* C_0 *of initial conditions such that*

$$(k^{(1)}(t)\,, k^{(2)}(t)) = (I_1 + I_2)t + C_0\,.$$

Proof: It is clear that

$$\frac{d(k^{(1)}(t)\,,\,k^{(2)}(t))}{dt} = \sum_{i=1}^{3} k_i^{(1)}(t)\,\frac{d\,k_i^{(2)}(t)}{dt} +$$

$$+\sum_{i=1}^{3}\frac{d\,k_i^{(1)}(t)}{dt}\cdot k_i^{(2)}(t) = |k^{(1)}|^2\cdot V^{(1)}(t) +$$

$$+\,|k^{(2)}(t)|^2\cdot V^{(2)}(t) = I_1 + I_2\,.$$

Lemma 3: *Let $I_1 \neq 0$, $I_2 \neq 0$. There exists another function C_1 of initial conditions such that*

$$|k^{(2)}(t)| = C_1\cdot|k^{(1)}(t)|^{\frac{I_1}{I_2}}\,.$$

Proof: It is easy to see that $\dfrac{d\,k_i^{(1)}(t)}{dt} = k_i^{(2)}(t)\cdot\dfrac{I_2}{|k^{(2)}(t)|^2} = I_2\cdot\dfrac{\partial}{\partial\,k_2^{(2)}}\,\ell n|k^{(2)}(t)|$. Therefore

$$\sum_{i=1}^{3}\frac{d\,k_i^{(1)}(t)}{dt}\cdot\frac{d\,k_i^{(2)}(t)}{dt} = I_2\cdot\sum_{i=1}^{3}\frac{\partial\,\ell n|k^{(2)}(t)|}{\partial\,k_i^{(2)}}\cdot\frac{d\,k_i^{(2)}}{dt} =$$

$$= I_2\,\frac{d}{dt}\,\ell n\,|k^{(2)}(t)| = \frac{d}{dt}\,I_2\cdot|k^{(2)}(t)|$$

because I_2 is the first integral. Changing the indices 1 and 2 we have $\sum_{i=1}^{3}\dfrac{d\,k_i^{(1)}(t)}{dt}\cdot\dfrac{d\,k_i^{(2)}(t)}{dt} = \dfrac{d}{dt}\,(I_1\,\ell n|k^{(1)}(t)|)$. Therefore

$$\frac{d}{dt}\,(I_1\,\ell n|k^{(1)}(t)|) = \frac{d}{dt}\,(I_2\,\ell n|k^{(2)}(t)|)$$

and

$$|k^{(2)}(t)| = C_1\cdot|k^{(1)}(t)|^{\frac{I_1}{I_2}}\,.$$

Lemma is proven.

The behavior of solutions of the system (7) and (8) depends on the values of the first integrals I_1, I_2 and the length $|J|$ of the vector $|J|$. We shall consider different cases.

I. $|J| = 0$. In this case the vectors $k^{(1)}(t)$, $k^{(2)}(t)$ are proportional to each other, $k^{(1)}(t) = a(t)\,k^{(2)}(t)$, $a(t) \neq 0$ and $V^{(1)}(t) = a^{-1}(t)$, $(k^{(1)}(t), v^{(1)}(t)) = 0$. Therefore, $k^{(2)}(t) = k^{(2)}(0)$, $v^{(1)}(t) = v^{(1)}(0)$. In the same way, $V^{(2)}(t) = a(t)(k^{(2)}(t), v^{(2)}(t)) = 0$ and $v^{(2)}(t) = v^{(2)}(0)$, $k^{(2)}(t) = k^{(2)}(0)$.

II. $|J| \neq 0$, $I_1 = I_2 = 0$. In this case $V^{(1)}(t) = V^{(2)}(t) = 0$, $v^{(1)}(t) = v^{(1)}(0)$, $v^{(2)}(t) = v^{(2)}(0)$, $k^{(1)}(t) = k^{(1)}(0)$, $k^{(2)}(t) = k^{(2)}(0)$.

III. $|J| \neq 0$, $I_1 \neq 0$, $I_2 = 0$. In this case $V^{(2)}(t) = 0$ and therefore $v^{(2)}(t) = v^{(2)}(0)$, $k^{(1)}(t) = k^{(1)}(0)$, $V^{(1)}(t) = \frac{I_1}{|k^{(1)}(0)|^2}$. From Lemmas 2 and 3

$$|k^{(1)}(t)|^2 \cdot |k^{(2)}(t)|^2 = (k^{(1)}(t)\,,\, k^{(2)}(t))^2 + |[k^{(1)}(t),\, k^{(2)}(t)]|^2 = (I_1 t + C_0)^2 + |J|^2\,.$$

Therefore $|k^{(2)}(t)| = |k^{(1)}(0)|^{-1}\,((I_1 t + C_0)^2 + |J|^2)$ and $|k^{(2)}(t)| \longrightarrow \infty$ as $t \longrightarrow \infty$. Denote by $\varphi(t)$ the angle between $k^{(2)}(t)$ and $k^{(1)}(t)$. We can write $|ctg\varphi(t)| = \frac{I_1 t + C_0}{|J|}$. This shows that $|k^{(2)}(t)| \longrightarrow \infty$ and its direction tends to the direction of $k^{(1)}(0)$ if $I_1 > 0$ or $-k^{(1)}(0)$ if $I_1 < 0$.

For the components of the vector $v^{(1)}(t)$ we have the equation

$$\frac{dv_i^{(1)}(t)}{dt} = v_i^{(2)}(t)\,V^{(1)}(t) = v_i^{(2)}(0) \cdot \frac{I_1}{|k^{(1)}(0)|^2}$$

From the last equation it follows that $v^{(1)}(t)$ grows linearly in time in the direction of $v^{(2)}(0)$ if $I_1 > 0$ and $-v^{(2)}(0)$ if $I_1 < 0$.

IV. $|J| \neq 0$, $I_1 = 0$, $I_2 \neq 0$.

This case is reduced to the previous one if we interchange the numbers of the particles.

V. $|J| \neq 0$, $I_1 = -I_2 \neq 0$.

In this case $(k^{(1)}(t), k^{(2)}(t)) = C_0$, $|k^{(1)}(t)|^2 \cdot |k^{(2)}(t)|^2 = C_1$. The angle between $k^{(1)}(t)$, $k^{(2)}(t)$ does not depend on t because $\cos\varphi(t) = \frac{C_0}{\sqrt{C_1}}$. Also

$V^{(2)}(t) \; = \; \frac{I_2}{|k^{(2)}(t)|^2} \; = \; \frac{I_2 \cdot |k^{(1)}|^2}{C_1}$. Therefore

$$\begin{aligned} \frac{d\,|k^{(1)}(t)|^2}{dt} &= 2(k^{(1)}(t)\,,k^{(2)}(t))\,V^{(2)}(t) = \\ &= \frac{2 \cdot C_0 \cdot I_2}{C_1} \cdot |k^{(2)}(t)|^2 \end{aligned}$$

and

$$|k^{(1)}(t)|^2 \; = \; |\,k^{(1)}(0)|^2\, e^{\frac{2\,C_0\,I_2}{C_1}\,t}\,. \tag{9}$$

In the same way

$$|k^{(2)}(t)|^2 \; = \; |k^{(2)}(0)|^2\, e^{\frac{2\,C_0\,I_1}{C_1}\,t}\,. \tag{10}$$

Assume that $C_0\,I_2 > 0$. Then $C_0\,I_1 < 0$ and the formulas (9) and (10) show that

$$\begin{aligned} |k^{(1)}(t)|^2 &= |k^{(1)}(0)|^2\, e^{-C_2 t}\,, \\ |k^{(2)}(t)|^2 &= |k^{(2)}(0)|^2\, e^{C_2 t} \end{aligned}$$

where $C_2 \; = \; -\frac{2\,C_0\,I_2}{C_1}$. This gives $\lim\limits_{t\longrightarrow\infty} |k^{(1)}(t)| \; = \; 0$, $\lim\limits_{t\longrightarrow\infty} |k^{(2)}(t)| \; = \; \infty$.

The estimation of $|v^{(i)}(t)|\,, i = 1,2$ is based on a trick which we shall use also in the other remaining cases. The vectors $k^{(1)}(t)$, $k^{(2)}(t)$, J constitute a non-orthogonal basis in R^3 because $J \perp k^{(1)}(t)$, $J \perp k^{(2)}(t)$ and $k^{(1)}(t)$, $k^{(2)}(t)$ are linearly independent since $J = [k^{(1)}(t)\,, k^{(2)}(t)] \neq 0$. For each t we can write

$$v^{(i)}(t) \; = \; a_i(t)\,J \; + \; b_i(t)\,k^{(1)}(t) \; + \; c_i(t)\,k^{(2)}(t)\,, i \; = \; 1,2\,. \tag{11}$$

Taking inner products of both sides of (11) with $J, k^{(1)}(t)$, $k^{(2)}$ and using the incompressibility condition we get two systems of three linear equations for $a_1(t)$, $b_1(t)$, $c_1(t)$ and $a_2(t)$, $b_2(t)$, $c_2(t)$. Solving them we have the explicit expressions for all coefficients:

$$a_1(t) = \frac{(v^{(1)}(t), J)}{|J|^2}, \ b_1(t) = \frac{-2\, V^{(1)}(t)(k^{(1)}(t), k^{(2)}(t))}{|J|^2},$$
$$c_1(t) = \frac{2\, V^{(1)}(t)\, |k^{(1)}(t)|^2}{|J|^2} \tag{12'}$$

and

$$a_2(t) = \frac{(v^{(2)}(t)\, J)}{|J|^2}, b_2(t) = \frac{2\, V^{(2)}(t)\, |k^{(2)}(t)|^2}{|J|^2},$$
$$c_2(t) = \frac{-2\, V^{(2)}(t)\, (k^{(1)}(t)\, ,\, k^{(2)}(t))}{|J|^2} \tag{12''}$$

In deriving (12′), (12″) we used the formula $|k^{(1)}(t)|^2\, |k^{(2)}(t)|^2 - (k^{(1)}(t),\, k^{(2)}(t))^2 = |J|^2$.

From (12′), (12″) one gets the following system of $O\mathcal{D}E$

$$\frac{da_1(t)}{dt} = a_2(t)\, V^{(1)}(t)$$
$$\frac{d\, a_2(t)}{dt} = a_1(t)\, V^{(2)}(t) \tag{13}$$

Using the lemmas 1,2,3 we come to the final expressions for $b_i(t)$, $c_i(t)$, $i = 1, 2$, namely

$$b_1(t) = -\frac{2\, C_0 \, \cdot I_1}{|J|^2 \, \cdot |k^{(1)}(t)|^2} = -\frac{2\, C_0\, I_1}{|J|^2\, |k^{(1)}(0)|^2} \exp\{C_2 t\},$$
$$c_1(t) = \frac{2\, I_1}{|J|^2},$$
$$b_2(t) = \frac{2\, I_2}{|J|^2},$$
$$c_2(t) = -\frac{2\, C_0 \, \cdot I_2}{|J|^2\, |k^{(2)}(t)|^2} = -\frac{2\, C_0\, I_2}{|J|^2 \, \cdot |k^{(2)}(0)|^2} \exp\{-C_2 t\}.$$

The system (13) implies

$$\frac{d(a_1^2(t) + a_2^2(t))}{dt} = 2\, a_1(t)\, a_2(t)\, (V^{(1)}(t) + V^{(2)}(t))\,. \tag{14}$$

From (14)

$$-|V^{(1)}(t)| - |V^{(2)}(t)| \leq \frac{d\,\ell n(a_1^2(t) + a_2^2(t))}{dt} \leq |V^{(1)}(t)| + |V^{(2)}(t)| \tag{15}$$

If we substitute in (15) the explicit expressions for $V^{(1)}(t)$, $V^{(2)}(t)$ we get the inequalities

$$\exp\left\{-\int_0^t \left(\frac{|I_1|}{|k^{(1)}(t)|^2} + \frac{|I_2|}{|k^{(2)}(t)|^2}\right) dt\right\} \leq \frac{a_1^2(t) + a_2^2(t)}{a_1^2(0) + a_2^2(0)} \leq \exp\left\{\int_0^t \left(\frac{|I_1|}{|k^{(1)}(t)|^2} + \frac{|I_2|}{|k^{(2)}(t)|^2}\right) dt\right\}$$

One can derive an explicit asymptotics for $v^{(i)}(t)$ as $t \longrightarrow \infty$.

VI. $|J| \neq 0,\, I_1 \neq 0,\, I_2 \neq 0,\, I_1 + I_2 \neq 0$.

From Lemmas 1 and 2

$$\begin{aligned} |k^{(1)}(t)|^2 \cdot |k^{(2)}(t)|^2 &= ((k^{(1)}(t)\,, k^{(2)}(t))^2 + \\ + |[k_1(t)\,, k_2(t)]|^2 &= ((I_1 + I_2)\, t + C_0)^2 + |J|^2 \end{aligned} \tag{16}$$

and from Lemma 3

$$|k^{(1)}(t)|^2 = \left[C_1^{-2}\, F(t)\right]^{\frac{I_2}{2(I_1+I_2)}} \tag{17}$$

$$|k^{(2)}(t)|^2 = C_1 \left[C_1^{-2}\, F(t)\right]^{\frac{I_1}{2(I_1+I_2)}} \tag{18}$$

where $F(t)$ is the right-hand side of (16).

It can be easily seen from (17) and (18) that if I_1 and I_2 have the same sign then $\lim_{t\longrightarrow\infty} |k^{(1)}(t)| = \lim_{t\longrightarrow\infty} |k^{(2)}(t)| = \infty$. If their signs are different then the length of one of the vectors $k^{(i)}(t), i = 1, 2$ tends to zero while the other one tends to infinity. The angle $\varphi(t)$ between $k^{(1)}(t)$ and $k^{(2)}(t)$ tends to zero or π as $t \longrightarrow \infty$ depending on the sign of $I_1 + I_2$.

The analysis of the behavior of the vectors $v^{(1)}(t)$, $v^{(2)}(t)$ is based on the formulas (12). We have

$$\begin{aligned} b_1(t) &= -\frac{2\,I_1((I_1+I_2)\,t+C_0)}{|J|^2\,\left[C_1^{-2}((I_1+I_2)\,t+C_0)^2\,+\,|J|^2\right]^{\frac{I_2}{I_1+I_2}}} \\ c_1(t) &= \frac{2\,I_1}{|J|^2}\,, \\ b_2(t) &= \frac{2\,I_2}{|J|^2}\,, \\ c_2(t) &= -\frac{2\,I_2((I_1+I_2)\,t+C_0)}{|J|^2\,\left[C_1^{-2}((I_1+I_2)\,t+C_0)^2\,+\,|J|^2\right]^{\frac{I_1}{I_1+I_2}}}\,. \end{aligned}$$

Thus

$$\lim_{t\longrightarrow\infty}\frac{b_1(t)}{t^{\frac{I_1-I_2}{I_1+I_2}}}\;=\;\text{const}\,,\;\lim_{t\longrightarrow\infty}\frac{c_2(t)}{t^{\frac{I_2-I_1}{I_1+I_2}}}\;=\;\text{const}.$$

and

$$\lim_{t\longrightarrow\infty}|b_1(t)|\;=\;\infty\,,\lim_{t\longrightarrow\infty}c_2(t)\;=\;0\text{ if }\frac{I_1-I_2}{I_1+I_2}\;>\;0\,,$$

$$\lim_{t\longrightarrow\infty}b_1(t)\;=\;0\,,\lim\,|c_2(t)|\;=\;\infty\text{ if }\frac{I_1-I_2}{I_1+I_2}\;<\;0\,.$$

The estimates for $a_1(t)$, $a_2(t)$ follow from (14). In particular, they show that $a_1(t)$, $a_2(t)$ are finite for all $t>0$.

We summarize all possible types of behavior of solutions of (7) and (8) in the formulation of the next theorem which gives the main result of this paper.

Theorem 2: *For any initial conditions $k^{(i)}(0)\neq 0$, $v^{(i)}(0)$, $i=1,2$ satisfying the incompressibility condition (8) solutions of the system (7) exist for all $t>0$.*

The functions

$$J = [k^{(1)}(0)\,, k^{(2)}(0)]\,,\; I_1 = \frac{1}{2}\,|k^{(1)}(0)|^2\,\cdot\,(k^{(2)}(0)\,, v^{(1)}(0))\,,$$

$$I_2 = \frac{1}{2}\,|k^{(2)}(0)|^2\,\cdot\,(k^{(1)}(0)\,,\, v^{(2)}(0))\,.$$

are the first integrals of (7) and (8). Depending on their values solutions can have the following types of behavior as $t \longrightarrow \infty$*:*

1. $|J| = 0.$

   $v^{(1)}(t) = v^{(i)}(0)\,, k^{(i)}(t) = k^{(i)}(0)\,, i = 1, 2.$

2. $|J| \neq 0$, $I_1 = I_2 = 0.$

   $v^{(i)}(t) = v^{(i)}(0)\,, k^{(i)}(t) = k^{(i)}(0)\,, i = 1, 2.$

3. $|J| \neq 0$, $I_1 \neq 0$, $I_2 = 0.$

   $k^{(1)}(t) = k^{(1)}(0)$, $v^{(2)}(t) = v^{(2)}(0).$

   $\lim\limits_{t \longrightarrow \infty} |k^{(2)}(t)| = \infty$, $v^{(1)}(t) = v^{(2)}(0)\,\frac{I_1 \cdot t}{|k^{(1)}(0)|^2} + v^{(1)}(0)$

4. $|J| \neq 0$, $I_1 = 0$, $I_2 \neq 0.$

   This case is reduced to 3) if we interchange the numbers of the particles.

5. $|J| \neq 0$, $I_1 = -I_2 \neq 0.$

   In this case $(k^{(1)}(t), k^{(2)}(t)) = (k^{(1)}(0), k^{(2)}(0) = C_0.$

   If $C_0\, I_1 > 0$ then $|k^{(1)}(t)|$ decreases exponentially, $|k^{(2)}(t)|$ exponentially grows, $\lim\limits_{t \longrightarrow \infty} |v^{(1)}(t)| = \infty.$

   If $C_0\, I_1 < 0$ then $|k^{(2)}(t)|$ decreases exponentially, $|k^{(1)}(t)|$ grows exponentially, $\lim\limits_{t \longrightarrow \infty} |v^{(2)}(t)| = \infty.$

6. $|J| \neq 0$, $I_1 \neq 0$, $I_2 \neq 0$, $I_1 + I_2 \neq 0$.

   If $sgn\ I_1 = sgn\, I_2$ then $\lim_{t \longrightarrow \infty} |k^{(i)}(t)| = \infty$, $i = 1, 2$.

   If $sgn\ I_1 = -\, sgn\, I_2$ then the length of one of the vectors $k^{(i)}(t)$, $i = 1, 2$ tends to ∞ while the length of the other one tends to 0. The angle $\varphi(t)$ between $k^{(1)}(t)$, $k^{(2)}(t)$ tends to zero if $I_1 + I_2 > 0$ and to π if $I_1 + I_2 < 0$ when $t \longrightarrow \infty$.

Theorem 2 shows that the behavior of solutions of (7) and (8) can be different from the behavior of solutions of the similar system with $\nu > 0$.

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References

1. E.I. Dinaburg and Y.G. Sinai, "A Quasi-Linear Approximation for the $3\mathcal{D}$-Navier-Stokes System," *Moscow Mathematical Journal*, Vol. 1, N3, 381-388, (2001).
2. E.I Dinaburg, "New Finite-Dimensional Approximations of the $3\mathcal{D}$-Navier-Stokes System," Doklady of Russian Academy of Sciences. (In print).