Separating Solution of a Recurrent Equation

by

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§1. Introduction

Consider a sequence of real numbers given by a recurrent equation

$$\wedge_p = \frac{1}{p-1} \sum_{p_1=1}^{p-1} f\left(\frac{p_1}{p}\right) \wedge_{p_1} \cdot \wedge_{p-p_1} . \tag{1}$$

Here f is a C^3-function on $[0,1]$. The sequence $\wedge_p$ is defined as soon as $\wedge_1 = y$ is given. In this sense we shall write $\wedge_p(y)$ and assume $y \geq 0$.

If $y' = cy$ then $\wedge_p(y') = c^p \wedge_p (y)$. Therefore, if $\wedge_p(y) \longrightarrow \infty$ as $p \longrightarrow \infty$ and $c \geq 1$ then also $\wedge_p(y') \longrightarrow \infty$ as $p \longrightarrow \infty$. If $\wedge_p(y) \longrightarrow 0$ as $p \longrightarrow \infty$ and $0 < c < 1$ then $\wedge_p(y') \longrightarrow 0$ as $p \longrightarrow \infty$. This implies that the set of $y > 0$ for which $\wedge_p(y) \longrightarrow \infty$ as $p \longrightarrow \infty$ is an open semi-line (y^+, ∞) while the set of $y > 0$ for which $\wedge_p(y) \longrightarrow 0$ as $p \longrightarrow \infty$ is an interval $(0, y^-)$. It is a natural question whether $y^- = y^+ = y^{(0)}$ and $\wedge_p(y^{(0)}) \longrightarrow \mathsf{const}$ as $p \longrightarrow \infty$. As it is easy to understand, this constant must be equal to $(\int_0^1 f(\gamma)d\gamma)^{-1}$ and it is our first assumption that $\int_0^1 f(\gamma)\, d\gamma > 0$. Without any loss of generality it is enough to consider the case $\int_0^1 f(\gamma)\, d\gamma = 1$ because if $f^{(1)}(y) = K\, f(y)$ for some constant K then for the corresponding sequence $\wedge_p^{(1)}(y)$ we have $\wedge_p^{(1)}(y) = K^{-1} \wedge_p (y)$.

The above formulated question appeared in our joint paper with Dong Li (see [LS]) on short time singularities in complex-valued solutions of the 3-dimensional Navier-Stokes system on R^3. There we needed the positive answer for the particular case $f(\gamma) = 6\gamma^2 - 10\gamma + 4$. Each of us found his own proof of the needed statement but the proofs were different and required different assumptions concerning the function f. The proof given by Dong Li can be found in his paper [L]. Below I present my proof which uses some inductive process. Here are the main assumptions about the function f:

(1) $f \in C^3([0,1])$.

(2) $f(0)$ or $f(1)$ is zero. Without any loss of generality we can consider the case $f(1) = 0$.

(3) $\int_0^1 f(\gamma)\, d\gamma = 1$. As it was already explained, this is not a restriction because of the scaling properties of (1).

(4) Let $f_1(\gamma) = f(\gamma) + f(1-\gamma)$, $f_2(\gamma) = -f_1(\gamma) - \gamma f_1'(\gamma)$, $f_3(\gamma) = \frac{1}{\gamma^2} \int_0^1 x f_2(x)\, dx$. Then $\int_0^1 \frac{1}{\sqrt{\gamma}} f_3(\gamma)\, d\gamma \neq -1$.

(5) The last assumption concerns the initial part of our inductive process. It will be formulated later in §2, §4.

Main Theorem. *If the conditions* (1)-(5) *are fulfilled, then there exists* $y^{(0)} > 0$ *such that for* $p \longrightarrow \infty$

$$\wedge_p(y^{(0)}) \longrightarrow \left(\int_0^1 f(\gamma)\, d\gamma \right)^{-1} = 1\,.$$

Clearly $y^{(0)}$ is unique.

We shall call $\wedge_p(y^{(0)})$ a separating solution of (1).

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§2. Derivation of the Main Recurrent Formula

It follows from (1) that

$$\wedge_p((1\,+\,x)y) \,=\, (1\,+\,x)^p \,\wedge_p(y)$$

and therefore

$$\frac{d\wedge_p((1\,+\,x)y)}{dx} \,=\, p(1\,+\,O(px))\,\wedge_p(y)\,.$$

We shall use this formula in the cases where $y = O(1)$, $\wedge_p(y) = O(1)$, $px = o(1)$. Then in the main order of magnitude $\frac{d}{dx}\wedge_p((1\,+\,x)y) \,=\, p\,\wedge_p(y)$ and does not depend on x, i.e. in this approximation $\wedge_p((1\,+\,x)y)$ is a linear function of x. We shall formulate this statement as a separate lemma.

Lemma 1. *Let for some numbers* A_1, A_2

$$A_1 \,\leq\, \wedge_p(y) \,\leq\, A_2\,.$$

Then there exists a constant C_1 depending on A_1, A_2 such that

$$\frac{d}{dx} \wedge_p (y + xy) = p(1 + \epsilon_p^{(1)}) \wedge_p (y) \tag{2}$$

and $|\epsilon_p^{(1)}| \leq C_1 \cdot p|x|$ provided that $p|x| \leq 1$.

Below in this text various absolute constants which appear during the proof will be denoted by A,B,C with indeces, various remainders will be denoted by ϵ, δ with indeces.

Choose some $\rho, 0 < \rho < 1$, and consider the sequence of intervals $\triangle_p = [1 - \frac{\rho}{\sqrt{p}}, 1 + \frac{\rho}{\sqrt{p}}]$. It is clear that $\triangle_p \supseteq \triangle_{p+1} \supseteq \ldots$. As was already mentioned, the proof of the main theorem is based on some inductive process. Assume that for all $q, 1 \leq q \leq p$ we have the intervals $\mathcal{D}_q = [a_q, b_q]$ on the y-axis such that

$$\wedge_q(a_q) = 1 - \frac{\rho}{\sqrt{q}}, \wedge_q(b_q) = 1 + \frac{\rho}{\sqrt{q}}$$

and $0 < A_3 < a_q \leq b_q \leq A_4$. Putting in (2) $y = a_p$ we get from Lemma 1 that the derivative $\frac{d}{dy} \wedge_q (y) = q(1 + \epsilon_q^{(2)})$ with $|\epsilon_q^{(2)}| \leq \frac{C_2}{\sqrt{q}}$ where C_2 depends only on ρ. In other words, if q is sufficiently large the function $\wedge_q(y)$ is strictly monotone and changes from $1 - \frac{\rho}{\sqrt{q}}$ till $1 + \frac{\rho}{\sqrt{q}}$ with the derivative of order of q. Now we can write

$$\wedge_{p+1}(a_{p+1}) - \wedge_p(a_p) = \frac{\rho}{\sqrt{p}} - \frac{\rho}{\sqrt{p+1}} = \frac{\rho}{2p^{3/2}} (1 + \epsilon_p^{(3)}), \; |\epsilon_p^{(3)}| \leq \frac{C_3}{p}.$$

Then

$$\frac{\rho}{2p^{3/2}} (1 + \epsilon_p^{(3)}) = \wedge_{p+1}(a_{p+1}) - \wedge_p(a_p) = \wedge_{p+1}(a_{p+1}) - \wedge_{p+1}(a_p)$$
$$+ \wedge_{p+1}(a_p) - \wedge_p(a_p) =$$
$$= (p+1)(1 + \epsilon_p^{(4)})(a_{p+1} - a_p) + \wedge_{p+1}(a_p) - \wedge_p(a_p)$$

with $|\epsilon_p^{(4)}| \leq \frac{C_4}{\sqrt{p}}$. This gives

$$(p+1)(1 + \epsilon_p^{(4)})(a_{p+1} - a_p) = \frac{\rho}{2p^{3/2}} (1 + \epsilon_p^{(3)}) - (\wedge_{p+1}(a_p) - \wedge_p(a_p)). \tag{3}$$

We shall use the recurrent equation (1) to find another expression for $\wedge_{p+1}(a_p) - \wedge_p(a_p)$. Put $\gamma = \frac{p_1}{p}$, $p_2 = p - p_1$, $\gamma' = \frac{p_1}{p+1}$. Then

$$\wedge_{p+1}(a_p) \;=\; \tfrac{1}{p} \sum_{p_1=1}^{p} f(\gamma') \; \wedge_{p_1}(a_p) \; \wedge_{p_2+1}(a_p) \;=$$

$$= \tfrac{1}{p} \sum_{p_1=1}^{p} f(\gamma')(\wedge_{p_1}(a_p)-1)(\wedge_{p_2+1}(a_p)-1)+$$

$$+\tfrac{1}{p} \sum_{p_1=1}^{p} f(\gamma')(\wedge_{p_1}(a_p)-1) \;+\; \tfrac{1}{p} \sum_{p_1=1}^{p} f(\gamma')(\wedge_{p_2+1}(a_p)-1)$$

$$-\tfrac{1}{p} \sum_{p_1=1}^{p} f(\gamma') \;=$$

$$= \tfrac{1}{p} \sum_{p_1=1}^{p} f(\gamma')(\wedge_{p_1}(a_p)-1)(\wedge_{p_2+1}(a_p)-1)+$$

$$+\tfrac{1}{p} \sum_{p_1=1}^{p} f_1(\gamma')(\wedge_{p_1}(a_p)-1) - \tfrac{1}{p} \sum_{p_1=1}^{p} f(\gamma')\,.$$

A similar formula can be written for $\wedge_p(a_p)$:

$$\wedge_p(a_p) \;=\; \tfrac{1}{p-1} \sum_{p_1=1}^{p-1} f(\gamma)\,(\wedge_{p_1}(a_p)-1)(\wedge_{p_2}(a_p)-1)+$$

$$+\tfrac{1}{p-1} \sum_{p_1=1}^{p-1} f_1(\gamma)(\wedge_{p_1}(a_p)-1) \;-\; \tfrac{1}{p-1} \sum_{p_1=1}^{p-1} f(\gamma)\,.$$

Subtracting $\wedge_p(a_p)$ from $\wedge_{p+1}(a_p)$ we get

$$\wedge_{p+1}(a_p) - \wedge_p(a_p) \;=\; \tfrac{1}{p}\, f\left(\tfrac{p}{p+1}\right)(\wedge_p(a_p)-1)(\wedge_1(a_p)-1)$$

$$+ \sum_{p_1=1}^{p-1} \left(\tfrac{1}{p}\, f(\gamma') - \tfrac{1}{p-1}\, f(\gamma)\right)(\wedge_{p_1}(a_p)-1)(\wedge_{p_2}(a_p)-1)$$

$$+\tfrac{1}{p} \sum_{p_1=1}^{p-1} f(\gamma')(\wedge_{p_1}(a_p)-1)(\wedge_{p_2'+1}(a_p) - \wedge_{p_2'}(a_p))$$

$$+\tfrac{1}{p}\, f_1\left(\tfrac{p}{p+1}\right)(\wedge_{p-1}(a_p)-1) \;+\; \sum_{p_1=1}^{p-1} \left(\tfrac{1}{p}\, f_1(\gamma') - \tfrac{1}{p-1}\, f(\gamma)\right)\,.$$

$$(\wedge_{p_1}(a_p)-1) \;+\; \tfrac{1}{p}\, f\left(\tfrac{p}{p+1}\right) - \sum_{p_1=1}^{p-1} \left(\tfrac{1}{p}\, f(\gamma') \;-\; \tfrac{1}{p-1}\, f(\gamma)\right) \;=\; \sum_{j=1}^{7} I_p^{(j)}\,.$$

Here $p_2' = p+1-p_1$. Each term will be estimated in the next section.

Now we can explain in detail the property (5) of the function f:

take some ρ, A_3, A_4. Then for some $p_0 = p_0(\rho, A_3, A_4)$ and all $q, , 1 < q \le p_0$ the segments $[a_q, b_q]$ for which $\wedge_q(a_q) = 1 - \frac{\rho}{\sqrt{q}}, \wedge_q(b_q) = 1 + \frac{\rho}{\sqrt{q}}, A_3 \le a_q \le b_q \le A_4$ exist.

Fix some number $\alpha > 0$ which later will be assumed to be sufficiently small and consider the sequence $p^{(n)}$, $p^{(n+1)} = (1+\alpha)p^{(n)}$, $n \ge 0$. The choice of α will be discussed in §4.

For each $n \ge 0$ we write $a_{p^{(n)}+1} - a_{p^{(n)}} = \frac{r^{(n)}}{(p^{(n)})^5/2}$ and this equality will be used for the definition of $r^{(n)}$. For $p^{(n)} < p \le p^{(n+1)}$ we write $a_{p^{(n)}+1} - a_{p^{(n)}} = \frac{r^{(n)}+\delta_p^{(n)}}{p^{5/2}}$. Denote $M_p = \max\limits_{q \le p} |r^{(n)} + \delta_q^{(n)}| = \max |a_q - a_{q-1}| q^{5/2}$. It is clear that M_p does not depend on the choice of $r^{(n)}$ and $\delta_p^{(n)}$ because $r^{(n)} + \delta^{(n)}$ does not depend on the n. Then M_p will be the main p numbers which we shall estimate below.

§3. Estimates of $I_p^{(j)}$

In this and the next section we consider p, $p^{(n)} < p \le p^{n+1)}$. First we consider the largest terms among $I_p^{(j)}$.

3.1 We start with

$$I_p^{(4)} = \frac{1}{p} f_1\left(\frac{p}{p-1}\right)(\wedge_{p-1}(a_p) - 1).$$

We have

$$\begin{aligned} I_p^{(4)} &= \frac{1}{p} f_1\left(\frac{p}{p-1}\right)(\wedge_{p-1}(a_{p-1}) - 1 + \wedge_{p-1}(a_p) - \wedge_{p-1}(a_{p-1})) \\ &= -\frac{\rho}{p^{3/2}} f_1(1) + \frac{1}{p} f_1(1)(\wedge_{p-1}(a_p) - \wedge_{p-1}(a_{p-1})) + \epsilon_p^{(5)} \end{aligned}$$

where $|\epsilon_p^{(5)}| \le \frac{C_5}{p^{5/2}}$. From (2)

$$\frac{1}{p}(\wedge_{p-1}(a_p) - \wedge_{p-1}(a_{p-1})) = (a_p - a_{p-1}) \cdot (1 + \epsilon_p^{(6)}),$$

$|\epsilon_p^{(6)}| \le \frac{C_6}{p^{1/2}}$. From the estimate $|a_p - a_{p-1}| \le \frac{M_p}{p^{5/2}}$ it follows that

$$\frac{1}{p}|\wedge_{p-1}(a_p) - \wedge_{p-1}(a_{p-1})| \le \frac{M_p}{p^{5/2}}\left(1 + \frac{C_6}{p^{1/2}}\right).$$

which gives

$$I_p^{(4)} = -\frac{\rho}{p^{3/2}} \cdot f_1(1) + \epsilon_p^{(7)}$$

and $|\epsilon_p^{(7)}| \le \frac{C_7\, M_p}{p^{5/2}}$. Later it will be shown that M_p are uniformly bounded. Consider

$$I_p^{(5)} = \sum_{p_1=1}^{p-1} \left(\frac{1}{p}\, f_1(\gamma') - \frac{1}{p-1}\, f_1(\gamma)\right) (\wedge_{p_1}(a_p) - 1)\,.$$

We can write

$$\frac{1}{p}\, f_1(\gamma') - \frac{1}{p-1}\, f_1(\gamma) = -\frac{\gamma' f_1'(\gamma') + f_1(\gamma')}{p(p-1)} + \epsilon_p^{(8)} = -\frac{f_2(\gamma')}{p(p-1)} + \epsilon_p^{(8)}\,,$$

$|\epsilon_p^{(8)}| \le \frac{C_8}{p^3}$. For the difference $\wedge_{p_1}(a_p) - 1$ from Lemma 1 and 2 we have

$$\wedge_{p_1}(a_p) - 1 = \wedge_{p_1}(a_{p_1}) - 1 + \wedge_{p_1}(a_p) - \wedge_{p_1}(a_{p_1}) = -\frac{\rho}{p_1^{1/2}} + (p_1 + \epsilon_{p_1}^{(1)})(a_p - a_{p_1})\,.$$

The estimate for $\epsilon_{p_1}^{(1)}$ was given before. Then

$$I_p^{(5)} = -\sum_{p_1=1}^{p-1} \frac{f_2(\gamma')}{p(p-1)} \cdot \frac{\rho}{p_1^{1/2}} + \sum_{p_1=1}^{p-1} \frac{f_2(\gamma')\, p_1(a_p - a_{p_1})}{p(p-1)} + \epsilon_p^{(3)}\,, \tag{4}$$

$|\epsilon_p^{(3)}| \le \frac{C_3}{p^2}$ and

$$\sum \frac{f_2(\gamma'}{p(p-1)} \cdot \frac{\rho}{p_1^{1/2}} = \frac{\rho}{p^{3/2}} \int_0^1 \frac{f_2(\gamma)\, d\gamma}{\sqrt{\gamma}} + \epsilon_p^{(10)}\,,$$

$|\epsilon_p^{(10)}| \le \frac{C_{10}}{p^{5/2}}$. For the second term in (4) we write

$$\begin{aligned}
J_p^{(5)} &= \sum_{p_1=1}^{p-1} \tfrac{f_2(\gamma')\, p_1(a_p - a_{p_1})}{p(p-1)} = \tfrac{1}{p-1} \sum_{p_1=1}^{p-1} f_2(\gamma') \cdot \gamma \sum_{p_1=1}^{p-1} f_2(\gamma') \cdot \gamma \sum_{q\,=\,p_1+1}^{p} (a_q - a_{q-1}) = \\
&= \tfrac{1}{p-1} \sum_{q_2=2}^{p} (a_q - a_{q-1}) \cdot \sum_{\gamma\,=\,\frac{p_1}{p}\,\le\,\frac{q}{p}} f_2(\gamma') \cdot \gamma = \\
&= \sum_{q=2}^{p} (a_q - a_{q-1}) \cdot \tfrac{1}{p-1} \sum_{p_1 \le q} f_2(\gamma') \cdot \gamma = \\
&= \sum_{q=2}^{p} (a_q - a_{q-1}) \left(\int_0^{q/p} f_2(\gamma) \cdot \gamma\, d\gamma + \epsilon_q^{(11)} \right) = \\
&= \sum_{q=2}^{p} (a_q - a_{q-1}) \cdot \left(\tfrac{q^2}{p^2} \cdot f_3\left(\tfrac{q}{p}\right) + \epsilon_q^{(11)} \right)
\end{aligned}$$

where $|\epsilon_q^{(11)}| \le \frac{C_{11}}{p}$. Recall now that $a_q - a_{q-1} = \frac{r^{(n)} + \delta_q^{(n)}}{q^{5/2}}$.

Therefore

$$J_p^{(5)} = \frac{r^{(n)}}{p^{3/2}} \sum_{q=2}^{p} \sqrt{\frac{p}{q}}\, f_3\left(\frac{q}{p}\right) \cdot \frac{1}{p} + \frac{1}{p^2} \sum_{q=2}^{p} \frac{1}{\sqrt{q}}\, f_3\left(\frac{q}{p}\right) \delta_q^{(n)} + \epsilon_p^{(12)} =$$

$$= \frac{r^{(n)}}{p^{3/2}} \int_0^1 \frac{f_3(\gamma)}{\sqrt{\gamma}}\, d\gamma + \frac{1}{p^2} \sum_{q=2}^{p} \frac{1}{\sqrt{q}}\, f_3\left(\frac{q}{p}\right) \delta_q^{(n)} + \epsilon_p^{(13)},$$

$|\epsilon_p^{(13)}| \le \frac{C_{13}}{p^{5/2}} R^{(n)}$ where $R^{(n)} = \max\limits_{m \le n} |r^{(m)}|$. The last sum will play an important role in the next section. Finally we have

$$I_p^{(5)} = -\frac{\rho}{p^{3/2}} \cdot \int_0^1 \frac{f_2(\gamma) d\gamma}{\sqrt{\gamma}} + \frac{r^{(n)}}{p^{3/2}} \int_0^1 \frac{f_3(\gamma)}{\sqrt{\gamma}}\, d\gamma + \frac{1}{p^2} \sum_{q=2}^{p} \frac{1}{\sqrt{q}}\, f_3\left(\frac{q}{p}\right) \delta_q^{(n)} + \epsilon_p^{(14)}$$

and $|\epsilon_p^{(14)}| \le \frac{C_{14}(1+R^{(n)})}{p^2}$.

3.2. In this part of §3 we shall consider other $I_p^{(j)}$, $j \ne 4, 5$ about which we shall show that they have a higher order of smallness and will be included later in the remainders. For $I_p^{(7)}$ we have

$$\begin{aligned} I_p^{(7)} &= -\sum_{p_1=1}^{p-2} \left(\tfrac{1}{p} f(\gamma') - \tfrac{1}{p-1} f(\gamma) \right) = \\ &= -\sum_{p_1=1}^{p-2} \left[\left(\tfrac{1}{p} f\left(\tfrac{p_1}{p+1}\right) - \tfrac{1}{p-1} f\left(\tfrac{p_1}{p+1}\right) \right) + \tfrac{1}{p-1} \left(f\left(\tfrac{p_1}{p+1}\right) - f\left(\tfrac{p_1}{p}\right) \right) \right] \\ &= \tfrac{1}{p(p-1)} \cdot \sum_{p_1=1}^{p-2} \left[f\left(\tfrac{p_1}{p+1}\right) + \tfrac{p_1}{(p+1)} f'\left(\tfrac{p_1}{p+1}\right) \right] + \epsilon_p^{(15)} \\ &= \tfrac{(p+1)}{p(p-1)} \cdot \int_0^1 [f(\gamma) + \gamma f'(\gamma)]\, d\gamma + \epsilon_p^{(16)} \end{aligned}$$

and $|\epsilon_p^{(16)}| \le \frac{C_{16}}{p^2}$. The last integral is zero in view of the condition $f(1) = 0$.

For $I_p^{(6)}$ we can write

$$|I_p^{(16)}| \le \frac{C_{18}}{p^2}.$$

Consider

$$I_p^{(3)} = \frac{1}{p}\sum_{p_1=1}^{p-1} f(\gamma')(\wedge_{p_1}(a_p)-1)\,(\wedge_{p_2'+1}(a_p)-\wedge_{p_2'}(a_p))\,.$$

We have

$$\wedge_{p_1}(a_p)-1 = \wedge_{p_1}(a_{p_1})-1+\wedge_{p_1}(a_p)-\wedge_{p_1}(a_{p_1})\,.$$

By construction, $\wedge_{p_1}(a_{p_1})-1 = -\frac{\rho}{\sqrt{p_1}}$ and with the help of Lemma 1

$$\wedge_{p_1}(a_p) - \wedge_{p_1}(a_{p_1}) = (p_1+\epsilon_{p_1}^{(1)})\,(a_p-a_{p_1})\,.$$

For the difference $(a_p-a_{p_1})$ we write

$$\begin{aligned} |a_p-a_{p_1}| &\le \sum_{q=p_1+1}^{p} |a_q-a_{q-1}| \le M_p\cdot\sum_{q=p_1+1}^{p}\tfrac{1}{q^{5/2}} \le \\ &\le M_p\cdot C_{19}\cdot\int_{p_1+1}^{p}\tfrac{dq}{q^{5/2}} = M_p\cdot C_{19}\cdot\tfrac{3}{2}\left(\tfrac{1}{p_1^{3/2}}-\tfrac{1}{p^{3/2}}\right) = \\ &= M_p\cdot C_{19}\cdot\tfrac{3}{2}\cdot\tfrac{1}{p_1^{3/2}}\left(1-\left(1-\tfrac{p_2}{p}\right)^{3/2}\right) \le M_p\cdot C_{20}\cdot\tfrac{p_2}{p_1^{3/2}\cdot p}\,. \end{aligned} \tag{5}$$

Thus

$$|\wedge_{p_1}(a_p) - \wedge_{p_1}(a_{p_1})| \le M_p\cdot C_{21}\cdot\frac{(p-p_1)}{p_1^{1/2}\cdot p}\,. \tag{6}$$

In the same way

$$|a_p - a_{p_2}| \le M_p\cdot C_{20}\cdot\tfrac{p_1}{p_2^{3/2}\cdot p} \tag{5'}$$

Now we can estimate the difference $\wedge_{p_2'+1}(a_p)-\wedge_{p_2'}(a_p)$:

$$\begin{aligned} &\wedge_{p_2'+1}(a_p)-\wedge_{p_2'}(a_p) = \wedge_{p_2'+1}(a_{p_2'}) - \wedge_{p_2'}(a_{p_2'})+ \\ &+\wedge_{p_2'+1}(a_p)-\wedge_{p_2'+1}(a_{p_2'}) - \wedge_{p_2'}(a_p)+\wedge_{p_2'}(a_{p_2'})\,. \end{aligned}$$

From Lemma 1 and from the previous estimates (5), (5′), (6) it follows that

$$\begin{aligned} &|(\wedge_{p_2'+1}(a_p)-\wedge_{p_2'+1}(a_{p_2'})) - (\wedge_{p_2'}(a_p)-\wedge_{p_2'}(a_{p_2'}))| \le \\ &\le C_{21}\,|a_p-a_{p_2'}| \le M_p\cdot C_{22}\cdot\frac{p_1}{p_2^{3/2}\cdot p}\,. \end{aligned}$$

This yields

$$I_p^{(3)} = \frac{1}{p} \sum_{p_1=1}^{p-1} f(\gamma')\,(\wedge_{p_1}(a_p) - 1)\,(\wedge_{p_2'+1}(a_p) - \wedge_{p_2'}(a_p)) =$$

$$= \frac{1}{p} \sum_{p_1=1}^{p-1} f(\gamma') \left(-\frac{\rho}{\sqrt{p_1}} + \wedge_{p_1}(a_p) - \wedge_{p_1}(a_{p_1})\right) ((\wedge_{p_2'+1}(a_{p_2'}) - - \wedge_{p_2'}(a_{p_2'})) +$$

$$+ (\wedge_{p_2'+1}(a_{p_2'}) - \wedge_{p_2'}(a_{p_2'})) + (\wedge_{p_2'+1}(a_p) - \wedge_{p_2'+1}(a_{p_2'}) - \wedge_{p_2'}(a_p) + \wedge_{p_2'}(a_{p_2'}))$$

and from the previous estimates and the inequality $|f(\gamma')| \le C_{23}\,\frac{p_2}{p}$

$$|I_p^{(3)}| \le \frac{1}{p^2} \cdot C_{23} \sum [p_2 \left(\frac{\rho}{\sqrt{p_1}} + M_p \cdot C_{20} \cdot \frac{p_2}{p_1^{1/2} \cdot p}\right) \cdot$$

$$\cdot (M_p \cdot C_{24} \cdot \frac{1}{p_2^{3/2}} + M_p \cdot C_{20} \cdot \frac{p_1}{p_2^{3/2} \cdot p})] =$$

$$= \frac{C_{25} \cdot M_p}{p^2} [\sum_{p_1=1}^{p-1} \frac{\rho\, p_2}{\sqrt{p_1} \cdot p_2^{3/2}} + \sum_{p_1=1}^{p-1} \frac{\rho\, p_2}{\sqrt{p_1}} \cdot \frac{p_1}{p_2^{3/2} \cdot p} +$$

$$+ \sum_{p_1=1}^{p-1} \frac{p_2^2}{p_1^{1/2} \cdot p} \cdot \frac{1}{p_2^{3/2}} + \sum M_p^2\, C_{20}^2 \cdot \frac{p_2^2}{p_1^{1/2} p} \cdot \frac{p_1}{p_2^{3/2} \cdot p}] \le \frac{(M_p+1)^2 \cdot C_{26}}{p^2}\,.$$

It will follow from our proof in §4 that M_p are uniformly bounded. Therefore $I_p^{(3)}$ has a higher order of smallness.

Next we estimate

$$I_p^{(2)} = \sum_{p_1=1}^{p-1} \left(\frac{1}{p}\, f(\gamma') - \frac{1}{p-1}\, f(\gamma)\right) (\wedge_{p_1}(a_p) - 1)\,(\wedge_{p_2}(a_p) - 1)\,.$$

As before,

$$|\wedge_{p_1}(a_p) - 1| \le |\wedge_{p_1}(a_{p_1}) - 1| + |\wedge_{p_1}(a_{p_1}) - \wedge_{p_1}(a_p)| \le \frac{\rho}{\sqrt{p_1}} + M_p \cdot C_{20} \cdot \frac{p_2}{p_1^{1/2} p}$$

and similarly

$$|\wedge_{p_2}(a_p) - 1| \le \frac{\rho}{\sqrt{p_2}} + M_p \cdot C_{20}\, \frac{p_1}{p_2^{1/2} \cdot p}\,.$$

Also, $|\frac{1}{p}\, f(\gamma') - \frac{1}{p-1}\, f(\gamma)| \le \frac{C_{27}}{p^2}$. Thus

$$I_p^{(2)} \le \frac{C_{27}}{p^2} \sum_{p_1=1}^{p-1} \left(\frac{\rho}{\sqrt{p_1}} + M_p \cdot C_{20} \cdot \frac{p_2}{p_1^{1/2} p}\right) \left(\frac{\rho}{\sqrt{p_2}} + M_p C_{20}\, \frac{p_1}{p_2^{1/2} p}\right)$$

$$= \tfrac{C_{27}\cdot\rho}{p^2} \sum_{p_1=1}^{p-1} \tfrac{1}{\sqrt{p_1}} \cdot \tfrac{1}{\sqrt{p_2}} + \tfrac{C_{27}\cdot\rho}{p^2} M_p \cdot C_{20} \cdot \sum \tfrac{\sqrt{p_1}}{\sqrt{p_2}\cdot p} +$$

$$+ \tfrac{M_p\cdot C_{20}\cdot C_{27}\rho}{p^2} \sum_{p_1=1}^{p-1} \tfrac{\sqrt{p_2}}{\sqrt{p_1}} \cdot \tfrac{1}{p} + \tfrac{C_{27}\cdot C_{20}\cdot M_p}{p^2} \sum_{p_1=1}^{p-1} \tfrac{\sqrt{p_2}\cdot\sqrt{p_1}}{p^2} \leq \tfrac{C_{28}(1+M_p)}{p^2}\,.$$

It remains to estimate

$$I_p^{(1)} = \frac{1}{p} f\left(\frac{p}{p+1}\right) \cdot (\wedge_p(a_p)-1)(\wedge_1(a_p)-1)\,.$$

It follows easily from the condition $f(1)=0$ that

$$|I_p^{(1)}| \leq \frac{C_{29}}{p^{5/2}}\,.$$

Now we can formulate the final result of all previous estimates.

$$\begin{aligned} \wedge_{p+1}(a_p - \wedge_p(a_p) &= -\tfrac{\rho}{p^{3/2}} f_1(1) - \tfrac{\rho}{p^{3/2}} \cdot \int_0^1 \tfrac{f_2(\gamma)}{\sqrt{\gamma}}\,d\gamma \\ &+ \tfrac{r^{(n)}}{p^{3/2}} \int_0^1 \tfrac{f_3(\gamma)}{\sqrt{\gamma}}\,d\gamma + \tfrac{1}{p^2} \sum_{q=2}^{p} \tfrac{1}{\sqrt{q}} f_3\left(\tfrac{q}{p}\right) \delta_q^{(n)} + \epsilon_p \end{aligned} \tag{7}$$

and $|\epsilon_p| \leq \frac{C_{30}\ln p}{p^2}\,(\max(1+M_p+R^{(n)}))^2$.

§4. The End of the Proof of the Main Theorem

As was mentioned before, the proof of the main theorem is based on induction. The possibility of the first $p^{(0)}$ steps is guaranteed by the property 5 of the function f. At the n^{th} step of the induction we consider $p > p^{(n)}$ and we have $a_{p+1} - a_p = \frac{r^{(n)}+\delta_p^{(n)}}{p^{5/2}}$. From (3)

$$a_{p+1} - a_p = \frac{\rho}{2p^{5/2}} - \frac{1}{p}(\wedge_{p+1}(a_p) - \wedge_p(a_p)) + \beta_p^{(1)}$$

where $|\beta_p^{(1)}| \leq \frac{B^{(1)}(\wedge_{p+1}(a_p)-\wedge_p(a_p))}{p^{3/2}}$. In this inequality $B^{(1)}$ is an absolute constant. From (7)

$$\begin{aligned} \wedge_{p+1}(a_p) - \wedge_p(a_p) &= -\tfrac{\rho}{p^{3/2}} f_1(1) + \tfrac{\rho}{p^{3/2}} \int_0^1 \tfrac{f_2(\gamma)}{\sqrt{\gamma}}\,d\gamma \\ &+ \tfrac{r^{(n-1)}}{p^{3/2}} \int_0^1 \tfrac{f_3(\gamma)}{\sqrt{\gamma}}\,d\gamma + \tfrac{1}{p^2} \sum_{q=2}^{p} \tfrac{1}{\sqrt{q}} f_3\left(\tfrac{q}{p}\right) \delta_q^{(n-1)} + \epsilon_p\,, |\epsilon_p| \leq \tfrac{C_{30}(1+M_p+R^{(n)})^2\ln p}{p^2}\,. \end{aligned} \tag{8}$$

Thus

$$r^{(n-1)} + \delta_{p+1}^{(n-1)} = \frac{\rho}{2} + \rho f_1(1) - \rho \int_0^1 \frac{f_2(\gamma)}{\sqrt{\gamma}} d\gamma - r^{(n-1)} \int_0^1 \frac{f_3(\gamma)}{\sqrt{\gamma}} d\gamma$$
$$+ \frac{1}{\sqrt{p}} \sum_{q=2}^{p} \frac{1}{\sqrt{q}} f_3\left(\frac{q}{p}\right) \delta_q^{(n-1)} + \beta_p^{(1)} p^{5/2} + \epsilon_p \cdot p^{3/2} \tag{9}$$

In the last expression, take $p = p^{(n)}$. Then $r^{(n)} = r^{(n-1)} + \delta_{p^{(n)}+1}^{(n-1)}$. It determines our "renormalization" at the n^{th}-step. Clearly, $|r^{(n)} - r^{(n-1)}| = |\delta_{p^{(n)}+1}^{(n-1)}|$.

In all previous formulas, replace $r^{(n-1)}$ by $r^{(n)} - (r^{(n)} - r^{(n-1)})$ and $\delta_p^{(n-1)}$ by $\delta_p^{(n)} + r^{(n)} - r^{(n-1)}$. Then from (9)

$$r^{(n)} = \frac{\rho}{2} + \rho f_1(1) - \rho \int_0^1 \frac{f_2(\gamma)}{\sqrt{\gamma}} d\gamma - r^{(n)} \int_0^1 \frac{f_3(\gamma)}{\sqrt{\gamma}} d\gamma$$
$$+ \frac{1}{\sqrt{p^{(n)}}} \sum_{q=2}^{p^{(n)}} \frac{1}{\sqrt{q}} f_3\left(\frac{q}{p^{(n)}}\right) \delta_q^{(n)} + (r^{(n)} - r^{(n-1)}) \int_0^1 \frac{f_3(\gamma)}{\sqrt{\gamma}} d\gamma \tag{10}$$
$$+ \frac{r^{(n)} - r^{(n-1)}}{\sqrt{p^{(n)}}} \sum_{q=2}^{p^{(n)}} \frac{1}{\sqrt{q}} f_3\left(\frac{q}{p^{(n)}}\right) + \beta_{p^{(n)}}^{(1)} (p^{(n)})^{5/2} + \epsilon_{p^{(n)}} (p^{(n)})^{3/2}.$$

For $p^{(n)} + 1 \leq p \leq p^{(n+1)}$ we use the formula analogous to (9):

$$r^{(n)} + \delta_{p+1}^{(n)} = \frac{\rho}{2} + \rho f_1(1) - \rho \int_0^1 \frac{f_2(\gamma)}{\sqrt{\gamma}} d\gamma - r^{(n)} \int_0^1 \frac{f_3(\gamma)}{\sqrt{\gamma}} d\gamma$$
$$+ \frac{1}{\sqrt{p}} \sum_{q=2}^{p} \frac{1}{\sqrt{q}} f_3\left(\frac{q}{p}\right) \delta_q^{(n)} + \beta_p^{(1)} p^{5/2} + \epsilon_p \cdot p^{3/2}. \tag{11}$$

The substitution of $r^{(n)}$ from (10) gives

$$\delta_{p+1}^{(n)} = \frac{1}{\sqrt{p}} \sum_{q=2}^{p} \frac{1}{\sqrt{q}} f_3\left(\frac{q}{p}\right) \delta_q^{(n)} - \frac{1}{\sqrt{p^{(n)}}} \sum_{q=2}^{p^{(n)}} \frac{1}{\sqrt{q}} f_3\left(\frac{q}{p^{(n)}}\right) \delta_q^{(n)}$$
$$- \beta_{p^{(n)}}^{(1)} (p^{(n)})^{5/2} - \epsilon_{p^{(n)}} (p^{(n)})^{3/2} - (r^{(n)} - r^{(n-1)}) \int_0^1 \frac{f_3(\gamma)}{\sqrt{\gamma}} d\gamma$$

$$- \frac{(r^{(n)} - r^{(n-1)})}{\sqrt{p^{(n)}}} \sum_{q=2}^{p^{(n)}} \frac{1}{\sqrt{q}} f_3 \left(\frac{q}{p^{(n)}} \right) \delta_q^{(n)} + \beta_p^{(1)} p^{5/2} + \epsilon_p \cdot p^{3/2}$$

$$= \sum_{s=p^{(n)}+1}^{p} \left(\frac{1}{\sqrt{s+1}} \sum_{q=2}^{s+1} \frac{1}{\sqrt{q}} f_3 \left(\frac{q}{s+1} \right) \delta_q^{(n)} - \frac{1}{\sqrt{s}} \sum_{q=2}^{s} \frac{1}{\sqrt{q}} f_3 \left(\frac{q}{s} \right) \delta_q^{(n)} \right) + \beta_p^{(2)} \tag{12}$$

where $\beta_p^{(2)}$ is the sum of all remaining terms in the previous expression.

It is most important to estimate the differences

$$J_s = \frac{1}{\sqrt{s+1}} \sum_{q=2}^{s+1} \frac{1}{\sqrt{q}} f_3 \left(\frac{q}{s+1} \right) \delta_q^{(n)} - \frac{1}{\sqrt{s}} \sum_{q=2}^{s} \frac{1}{\sqrt{q}} f_3 \left(\frac{q}{s} \right) \delta_q^{(n)} .$$

Let $\triangle_p^{(n)} = \max\limits_{1<q\le p} |\delta_q^{(n)} \cdot q^{\frac{1}{4}}|$. Then

$$J_s = \frac{1}{\sqrt{s+1}} \cdot \frac{1}{\sqrt{s+1}} \cdot f_3(1)\, \delta_{s+1}^{(n)} + \left(\frac{1}{\sqrt{s+1}} - \frac{1}{\sqrt{s}} \right) \sum_{q=2}^{s} \frac{1}{\sqrt{q}} f_3 \left(\frac{q}{s+1} \right) \delta_q^{(n)}$$

$$+ \frac{1}{\sqrt{s}} \sum_{q=2}^{s} \frac{1}{\sqrt{q}} \left(f_3 \left(\frac{q}{s+1} \right) - f_3 \left(\frac{q}{s} \right) \right) \delta_q^{(n)} .$$

Direct estimates of each part of the last expression give

$$|J_s| \le \frac{C_{31}}{s^{1\frac{1}{4}}} \cdot \triangle_p^{(n)} + \frac{C_{32}\, \triangle_p^{(n)} \cdot s^{\frac{1}{4}}}{s^{3/2}} + \frac{C_{33} \cdot \triangle_p^{(n)}}{s^{1\frac{1}{4}}} = \frac{C_{34} \triangle_p^{(n)}}{s^{1\frac{1}{4}}} .$$

Therefore,

$$\left| \frac{1}{\sqrt{p}} \sum_{q=2}^{p} \frac{1}{\sqrt{q}} f_3 \left(\frac{q}{p} \right) \cdot \delta_q^{(n)} - \frac{1}{\sqrt{p^{(n)}}} \sum_{q=2}^{p^{(n)}} \frac{1}{\sqrt{q}} f_3 \left(\frac{q}{p^{(n)}} \right) \cdot \delta_q^{(n)} \right|$$

$$= \left| \sum_{s=p^{(n)}+1}^{p} J_s \right| \le C_{34} \cdot \triangle_p^{(n)} \cdot \sum_{p=p^{(n)}+1}^{p} \frac{1}{s^{1\frac{1}{4}}} \le C_{35} \frac{\triangle_p^{(n)} \cdot (p - p_n)}{(p^{(n)})^{1\frac{1}{4}}} . \tag{13}$$

From the estimates of §3

$$|\epsilon_{p^{(n)}}| \cdot (p^{(n)})^{3/2} \le \frac{C_{36}(1 + M_{p^{(n)}})^2}{(p^{(n)})^{5/11}}, |\epsilon_p| \cdot p^{3/2} \le \frac{C_{36}(1 + M_p)^2}{p^{5/11}} . \tag{14}$$

Instead of 5/11 we could take any power less than $\frac{1}{2}$. The value of the constant C_{34} depends on this power.

The estimate of $r^{(n)} - r^{(n-1)}$ is done with the help of (9) written for $p = p^{(n)}$ and (10):

$$|r^{(n)} - r^{(n-1)}| = |\delta^{(n-1)}_{p^{(n)}+1}| \leq \frac{\triangle^{(n)}_{p^{(n)}+1}}{p_n^{1/4}} . \tag{15}$$

If $\triangle_p^{(n)}$ are uniformly bounded then $|r^{(n)-r^{(n-1)}}|$ decay exponentially with n. We can write

$$|(r^{(n)} - r^{(n-1)}) \int_0^1 \frac{f_3(\gamma)}{\sqrt{\gamma}}\, d\gamma| \leq \frac{C_{37}\triangle^{(n)}_{p^{(n)}+1}}{p_n^{1/4}} = \frac{C_{37} \cdot \triangle^{(n)}_{p^{(n)}+1}}{p_0^{\frac{1}{4}}(1+\alpha)^n} \tag{16}$$

$$|(r^{(n)} - r^{(n-1)}) \frac{1}{\sqrt{p^{(n)}}} \sum_{q=2}^{p^{(n)}} \frac{1}{\sqrt{q}} f_3 \left(\frac{q}{p}\right)| \leq \frac{C_{37} \cdot \triangle^{(n)}_{p^{(n)}+1}}{p_n^{1/4}} = \frac{C_{37} \cdot \triangle^{(n)}_{p^{(n)}+1}}{p_0^{\frac{1}{4}}(1+\alpha)^n} \tag{17}$$

It remains to estimate $\beta_p^{(1)}\, p^{5/2}$. We have (see above)

$$|\beta_p^{(1)}| \leq \frac{B^{(1)}| \wedge_{p+1}(a_p) - \wedge_p(a_p)|}{p^{3/2}}$$

and from (8)

$$|\beta_p^{(1)}| \cdot p^{3/2} \leq B^{(1)}| \wedge_{p+1}(a_p) - \wedge_p(a_p)| \leq \frac{B^{(1)}}{p^{3/2}} (C_{37} + C_{38}\,\triangle_p^{(n)}) .$$

Returning back to (12), (13), (14), (15), (16), (17) we have

$$|\beta_p^{(2)}| \leq \frac{2B^{(1)}}{p^{1/2}} (C_{37} + C_{38}\,\triangle_p^{(n)}) + \frac{C_{36}(1 + M_p)^2}{p^{5/11}} \tag{18}$$

$$+ \frac{2C_{36} \cdot \triangle_p^{(n)}}{p_0^{\frac{1}{4}}(1+\alpha)^n} \leq \frac{2B^{(1)}C_{37}}{p^{1/2}} + \frac{C_{39}\,\triangle_p^{(n)}\,(1+M_p)^2}{p^{5/11}} .$$

Now come back to (12), (13), (15), (16), (17). We can write

$$|\delta_p^{(n)}| \leq C_{35} \frac{\triangle_p^{(n)}(p-p_n)}{(p^{(n)})^{1\frac{1}{4}}} + \frac{2B^{(1)}C_{37}}{p^{1/2}} + \frac{C_{39} \cdot \triangle_p^{(n)}(1+M_p)^2}{p^{5/11}} \tag{19}$$

This yields

$$\triangle_{p+1}^{(n)} \leq \frac{2B^{(1)}C_{37}}{p^{1/2}} + \frac{C_{35}\triangle_p^{(n)}(p-p_n)}{(p^{(n)})^{1\frac{1}{4}}} + \frac{C_{39}\,\triangle_p^{(n)}(1+M_p)^2}{p^{5/11}} . \tag{20}$$

Since $p - p_n \leq \alpha p_n$ we have

$$\triangle_{p+1}^{(n)} \leq \frac{2B^{(1)}C_{37}}{p^{1/2}} + \frac{\triangle_p^{(n)}(C_{35} + C_{39}(1 + M_p)^2)}{(p^{(n)})^{1/4}} \tag{21}$$

Also $|r^{(n)} - r^{(n-1)}| \leq \frac{\triangle_{p_n}^{(n)}}{p_n^{1/4}}$ and

$$M_{p+1} = \max_{q \leq p+1} |a_q - a_{q-1}|\, q^{5/2} \leq |r^{(n)}| + \max_{q \leq p+1} |\delta_q^{(n)}| \leq$$

$$\leq |r^{(1)}| + \sum_{m=2}^{n} |r^{(m)} - r^{(m-1)}| + \frac{\triangle_{p+1}^{(n)}}{(p+1)^{1/4}}\,.$$

From (21) it follows easily that $|\triangle_{p+1}^{(n)}| \leq \frac{C_{40}}{p^{1/2}}$ and it implies that $M_p \leq C_{41}$. Remind that C with an index is an absolute constant.

If p_0 is large enough then all $a_p \in \left[\frac{1}{2}A_3, 2A_4\right]$ (see the end of §2). Since the derivatives of all $\wedge_p(y)$ within this interval are close to 1 the points b_p also belong to the segment $\left[\frac{1}{2}A_3, 2A_4\right]$. The distance $b_p - a_p \leq \frac{p \cdot C_{42}}{p^{3/2}}$. Therefore, $\lim_{p \longrightarrow \infty} a_p = \lim_{p \longrightarrow \infty} b_p = y^{(0)}$. Theorem is proven.

References

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