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Renormalization Group Method
in
Probability Theory, Statistical Physics
and
Dynamical System
Applications to the
Problem of Blow Ups of Solutions
of the
$3-\mathcal{D}$ Navier-Stokes System

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The main content of these lectures is the exposition of our joint paper with *Dong Li* (IAS, Princeton, NJ)

> "Renormalization Group Method and Blow Ups of Complex Solutions of $3-\mathcal{D}$ Navier-Stokes System"

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where we show that there is an open set in the space of 10-parameter families of initial conditions so that for each family from this set there are values of parameters for which the solution develops a blow-up in finite time so that the energy and the enstrophy tend to infinity as time approaches the critical value.

We consider complex solutions which do not satisfy the energy inequality. The explanation of the number 10 and the whole strategy of the proof require Renormalization Group Method which is exposed in Part I of these notes. In Part II we describe simple results related to the problem of existence and uniqueness of solutions for the $3-\mathcal{D}$ Navier-Stokes System.

In Part III we describe our joint paper with *Dong Li*.

Part I.

The Main Part of these lectures is based on the Renormalization Group Method (RGM). Its idea can be seen from the following simple picture. Assume that we have a smooth diffeomorphism of a compact smooth manifold M, x_0 is a fixed point of f, i.e., $f(x_0) = x_0$. Linearize f near x_0 and denote by L the corresponding linear operator. In general, L has no additional structure except its spectrum and the linear space H where L acts can be decomposed onto two parts: $H = H^{(s)} + H^{(nu)}$. Each of the subspaces $H^{(s)}$, $H^{(nu)}$ is invariant under L and $\| L^m \|_{H^{(s)}} \leq C_1 \rho_s^m$, $\| L^m e \| \geq C_2(\rho_{nu})^m \| e \|$ for every $e \in H^{(nu)}$. The constants C_1, C_2 depend on the choice of the norm and $\rho_{nu} > \rho_s$. In the cases which we shall consider $\rho_s < 1$ while $\rho_{nu} = 1$. According to the well-known Hadamard-Perron theorem the fixed point x_0 has a stable manifold $\Gamma^{(s)}$ such that for any $y \in \Gamma^{(s)}$ the iterations $f^n y \longrightarrow x_0$ as $n \longrightarrow \infty$. Then codim $\Gamma^{(s)} = \dim H^{(nu)}$ and usually both are finite even in the infinite-dimensional setting. Take a local manifold F, $\dim F = \dim H^{(nu)}$ which is C^1- close to $H^{(nu)}$. Then it intersects $\Gamma^{(s)}$ at one point $x(F)$. It means that for some open set in the space of $\dim H^{(nu)}$-families of points for any family from this set one point y belongs to $\Gamma^{(s)}$. Therefore under the iterations of f it converges to x_0. In concrete situations this convergence implies important corollaries which are needed for a problem under consideration.

Renormalization Group Method works in many infinite-dimensional settings as well where $\dim H^{(nu)} < \infty$. Below we describe several examples.

1. RGM in probability theory. Not so many probabilists know that classical limit theorems for the sums of independent random variables can be proven with the help of RGM. We shall describe the simplest case. Consider a sequence of *iidrv* $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ whose distribution has a density $p(x)$. For simplicity we assume that $p(x)$ is an even function, $p(-x) = p(x)$, and $\int x^2 p(x)dx = 1$. Let $\zeta_m = \frac{\xi_1+\xi_2+\cdots+\xi_2m}{2^{m/2}}$. Then $\zeta_{m+1} = \frac{\zeta'_m+\zeta''_m}{\sqrt{2}}$ where ζ'_m, ζ''_m are independent random variables having the same distribution as ζ_m. If $p_m(x)$ is the density of distribution of ζ_m then

$$p_{m+1}(x) = \sqrt{2}\int_{-\infty}^{\infty} p_m(x\sqrt{2}-y)\, p_m(y)\, dy \tag{1}$$

The formula (1) shows that $p_{m+1} = f(p_m)$ where f is the quadratic operator given by the *rhs* of (1). Our goal is to study $p_m = f^m(p_1)$ as $m \longrightarrow \infty$. As the first step we find fixed points of f i.e., the densities q for which

$$q(x) = \sqrt{2}\int_{-\infty}^{\infty} q(x\sqrt{2}-y)\, q(y)dy \tag{2}$$

It is easy to check that the family of Gaussian densities $q_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma}}\exp\left\{-\frac{x^2}{2\sigma}\right\}$, $\sigma > 0$, is a one-parameter family of fixed points. The second step of RGM is the linearization of (2) near q_σ and the study of the spectrum of this linearization. The linear map L is given by the linear integral operator L_σ,

$$L_\sigma h(x) = 2\sqrt{2}\int_{-\infty}^{\infty} h(x\sqrt{2}-y)\, q_\sigma(y)dy\,.$$

Sometimes L_σ is called Gauss integral operator. Assume that $\sigma = 1$. Eigen-functions of L_1 are Hermite functions $He_m(x)\,\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2}\right\}$ where $He_m(x)$ is the m-th Hermite polynomial. The Hermite polynomials will play an important role later. The corresponding eigen-values are $\lambda_m = \frac{1}{2^{\frac{m}{2}-1}}$. Since we consider the space of even functions, m must be even. Since we consider small perturbations in the space of probability densities they cannot have projections to the zeroth eigen-vector with $m = 0$. Then $\lambda_2 = 1$ is a neutral eigen-value. This is connected with the fact that the second moment of our distribution is invariant under f, i.e.,

$$\int_{-\infty}^{\infty} x^2 f(p(x))dx = \int_{-\infty}^{\infty} x^2\, p(x)\, dx\,.$$

The remaining part of the spectrum is stable, i.e., $\lambda_m < 1$ if $m > 2$. Methods of non-linear dynamics allow to prove the following theorem.

Theorem 1. *Let* $p_1(x) = \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2}\right\}(1 + h(x))$ *where*

$$\int_{-\infty}^{\infty} h(x)\,\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2}\right\}dx = \int_{-\infty}^{\infty} x^2 h(x)\cdot\frac{1}{\sqrt{2\pi}}\exp\left\{\frac{-x^2}{2}\right\}dx = 0$$

and h is small in $L^2(R^1, \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2}\right\})$. *Then* $f^m(p_1) \longrightarrow \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2}\right\}$ *in the sense of this space* L^2.

The statement of the theorem is a local version of the central limit theorem. Probability theory has special tools to prove the global version of Theorem 1 and global stability of the Gaussian fixed point. However, non-linear methods allow to prove an analogous statement for functions p_1 which can take positive and negative values. As it will be seen later, RGM works in many other cases.

2. RGM in statistical mechanics. RGM was proposed for the analysis of phase transitions in the works of M. Fisher, L. Kadanoff and K. Wilson (see, e.g. [F], [K], [W]). Here I shall briefly describe the so-called Dyson hierarchical model where one can clearly see how does it work. Take a growing sequence of finite sets $V_n, |V_n| = 2^n$ and each set V_{n+1} is the union of two similar subsets $V_{n+1} = V_n' \cup V_n''$, $|V_n'| = |V_n''| = 2^n$. Consider

spin model on V_{n+1} where each spin variable $\sigma(x), x \in V_{n+1}$ takes values ± 1. The Hamiltonian of the system takes the form:

$$H(\sigma(V_{n+1})) = H(\sigma(V_n')) + H(\sigma(V_n'')) - \frac{c^n}{2^{2n}} \left(\sum_{x \in V^{(n+1)}} \sigma(x) \right)^2$$

where c is a parameter. The last term describes the interaction between $\sigma(V_n'), \sigma(V_n'')$. A special feature of the hierarchical model is that the interaction depends only on the total spin $\sigma(V^{(n+1)}) = \sum_{x \in V^{(n+1)}} \sigma(x)$. Write down the distribution of the total spin which follows from the Gibbs distribution:

$$p_n(t, \beta) = \frac{1}{Z_n} \sum_{\sigma(V_n): \frac{1}{2^n} \sum_{x \in V^{(n)}} \sigma(x) = t} \exp\{-\beta H(\sigma(V_n))\}, \; Z_n = \sum_{\sigma(V_n)} \exp\{-\beta H(\sigma(V_n))\}$$

where β is the inverse temperature. Then from the formula for the Hamiltonian it follows that

$$p_n(t, \beta) = \wedge_{n+1} \exp\left\{\beta c^n t^2\right\} \sum_{t' + t'' = t} p_{n-1}(t', \beta)\, p_{n-1}(t'', \beta) \tag{3}$$

The equation (3) resembles (1). The main problem here is to study the behavior of p_n as $n \longrightarrow \infty$ for the inverse critical temperature βcr. The advantage of the hierarchical model is the possibility to formulate explicitly the condition on βcr as the condition that the typical values t of $2^{-n}\, \sigma(V_n)$ are such that the interaction $c^n t^2 = O\,(1)$. This implies the equation for the fixed point of RGM:

$$q(x; \beta) = \exp\left\{\beta x^2\right\} \int_{-\infty}^{\infty} q\left(\frac{x}{\sqrt{c}} + u; \beta\right) q\left(\frac{x}{\sqrt{c}} - u; \beta\right) du \tag{4}$$

which is analogous to (1).

As in the previous example, (4) has a curve of Gaussian solutions $q(x;\beta) = \sqrt{\frac{\beta c}{r(2-c)}}$ $\exp\left\{-\frac{\beta c x^2}{2-c}\right\}$. Again, the next step is the analysis of stability of these fixed points. It turns out that the Gaussian fixed point is stable (in the sense of RGM) only if $\sqrt{2} < c < 2$. In our paper with Blekher (see [BS1]) we proved the convergence of the distributions $p_n\left(tc^{n/2}, \beta c r\right)$ to this point. At $c = \sqrt{2}$ some bifurcation takes place and the Gaussian point becomes unstable. It gets replaced by another fixed point which decays faster than Gaussian and is stable. This non-Gaussian point was constructed in our other paper with Blekher (see [BS2]). A very good exposition of related results can be found in the book [CE] by Collet and Eckmann. More recent results here were obtained by H. Koch and Wittwer (see [KW]).

The original motivation of the works by Fisher, Kadanoff and Wilson was the analysis of the critical points in more realistic lattice models of statistical mechanics. This leads to the theory of limit theorems of probability theory for sums of strongly dependent random variables. This theory is still waiting for its development. However, some basic definitions can be given. Namely, consider a stationary random field $\{\xi(n), n \in \mathbb{Z}^d\}$ on the d-dimensional lattice $\mathbb{Z}^d$. Its probability distribution is denoted by P. Then $\xi'(n) = \frac{1}{d^\gamma}\sum_e \xi(2n+e)$ is again a stationary random field. Here $2n = (2n_1, 2n_2, \dots 2n_d)$ if $n = (n_1, n_2, \dots, n_d)$ and e is any d-dimensional vector whose components are 0 and 1. The transition from $\{\xi(n)\}$ to $\{\xi'(n)\}$ is called Kadanoff block-spin transformation. The probability distribution corresponding to $\{\xi'(n)\}$ is denoted by P'.

Definition 1. *Probability distribution P is called scale-invariant if $P' = P$.*

The scaling hypothesis in the theory of phase transitions says that only scale-invariant distributions can be limiting distributions of spin variables at the critical temperature. This condition of scale-invariance is an infinite-dimensional analog of the equation for the fixed point of RGM. Again, the next step is to find examples of scale invariant fixed points within the class of Gaussian stationary fields.

Theorem 2. *Let $f(\lambda_1, \dots \lambda_d)$ be an homogeneous positive function of degree $d(\alpha+1)$. Then the function*

$$\rho(\lambda_1 \dots \lambda_d) = \prod_{s=1}^{d} |e^{2\pi i \lambda_s} - 1|^2 \sum_{m \in \mathbb{Z}^d} \frac{1}{f(\lambda + m)}$$

is the spectral density of the scale-invariant Gaussian random field.

The case $f(\lambda_1, \ldots, \lambda_d) = \lambda_1^2 + \lambda_2^2 + \ldots + \lambda_d^2$, i.e., $\alpha = \frac{2}{d} - 1$, deserves a special attention. The Gibbs distribution corresponding to this field can be written as Const $\exp\{-\frac{1}{2}(\nabla\varphi, \nabla\varphi)\}$. Sometimes it is called Gaussian free field. The interaction in the Lagrangian of this field is short-ranged. Therefore it is natural to expect that this field can appear as the limiting distribution of the Ising-type model at the critical temperature. However, the Gaussian scale-invariant distribution is stable only if the dimension $d \geq 5$. The case $d = 4$ is marginal and requires a non-standard normalization. The corresponding statement was formulated in many papers by physicists (see, e.g. [PP], [Ka]). Mathematical results can be found in the works by [A], [Fr], [GK]. Let me mention also the paper by Pinson and Spencer [PS] where the authors proved the stability of the fixed point of the two-dimensional Ising model under even perturbations. The modern development of this topic is connected with the conformal field theory and Loewner stochastic equations.

One-dimensional Gaussin random field with $\alpha = 0$ appears in many problems of the theory of random matrices.

3. RGM in the theory of dynamical systems. For the first time RGM in the theory of dynamical systems appeared in the works of Feigenbaum (see [F1], [F2]) on universality in period-doubling bifurcations.

Assume that we have a family of one-dimensional maps $x \longrightarrow f(x; \lambda)$ depending on some parameter λ. A typical example is the quadratic map $x \longrightarrow \lambda x(1-x)$ and $0 \leq x \leq 1$. Let $x_0 = x_0(\lambda)$ be a stable fixed point for some λ_0, i.e., $|f'_x(x_0, \lambda_0)| < 1$. As λ increases there appears a value λ_1 such that $x_0(\lambda_1)$ looses its stability and $|f'_x(x_0(\lambda_1), \lambda_1)| = 1$. If $f'_x(x_0(\lambda_1), \lambda_1) = -1$ then typically a period-doubling bifurcation takes place, where the point $x_0(\lambda)$ becomes unstable and a new stable periodic point $x_1(\lambda)$ of period 2 arises.

As λ further increases, there appears the value λ_2 for which $x_1(\lambda_2)$ becomes unstable and gets replaced by the new point $x_2(\lambda_2)$ of period 4 which is stable in a small right semi-neighborhood of λ_2 and so on. Denote by λ_n the values of parameter where the subsequent bifurcations take place. The Feigenbaum universality says that typically λ_n converge to a limit λ_∞ and this convergence is exponential, i.e., $\lambda_n - \lambda_\infty \sim C(f)\alpha^n$ where the constant $C(f)$ depends on the family and α is the famous universal Feigenbaum constant, $\alpha^{-1} \sim 4.6992\ldots$. The basic idea of Feigenbaum to explain this universality was the use of RGM. Let $\bar{\lambda}_n$, $\lambda_n < \bar{\lambda}_n < \lambda_{n+1}$, be such that at the fixed point $x(\bar{\lambda}_n)$ the derivative of $f^{(2n)}$ is zero. The existence of $\bar{\lambda}_n$ is natural because this derivative changes from -1 until 1 as λ changes form λ_n to λ_{n+1}. Feigenbaum assumed that the form of $f^{(2^n)}(\cdot\,;\,\bar{\lambda}_n)$ is universal, i.e., it does not depend on the family $f(\cdot, \lambda)$. If so then the form of the universal function ψ must satisfy the functional equation

$$\psi(x) \;=\; -\theta\psi(\psi(\theta^{-1}x))\,,\; \theta \;=\; -\,\frac{1}{\psi(1)}\,.$$

Feigenbaum found ψ numerically and through it derived the constant α. There were many mathematical papers (Coullet, Tresser [CT], Derrida, Gervais, Pomeau [DGP], Collet, Eckmann, Lanford [CEL], Lanford [L], the works by $\mathcal{D}$. Sullivan [Su], the book by de Melo and van Strien [deMvS] and others) where the authors proved all necessary steps of RGM in Feigenbaum universality.

After the works of Feigenbaum and others RGM became very popular in dynamics. Khanin and I applied it to the Arnold problem about rectifying circle maps. Let $f(x)$ be such that f is monotone continuous and $f(x+1) = f(x)+1$. Then we can consider the homeomorphism of the circle $x \xrightarrow{\varphi} \{f(x)\}$. It follows from Denjoy theory that if $f \in C^1$ and the rotation number is irrational then there exists the change of variables $y = \chi\,(x)$ such that in new variable y the homeomorphism φ is reduced to the rotation, $\varphi(y) = y + \rho$. Arnold problem was to study the smoothness of χ as function of φ. It was solved completly by M. Herman (see [H]) and J-C. Yoccoz (see [Y]).

We found in [KS-1] another way to prove the results of [H] and [Y] which in some cases gives sharper results. Denote by q_n the denominator of the n-th approximant of the rotation number of φ. We consider φ^{q_n} on intervals of the length $O(\overset{\perp}{q_n})$. The basic idea was to show that the rescaled map asymptotically becomes linear. The linear map is the fixed point of the corresponding Renormalization Group which has enough stability to prove the convergence to this point. The paper by Khanin and Teplitsky [KT] gives the complete description of the corresponding technique.

It turns out that the KAM-theory of 2-dimensional twist maps can be also exposed as a problem of RGM (see [KS-2]). Interesting results concerning the so-called critical KAM-curves were obtained by R. Mackay [MK]. He also used the RGM-method.

Concluding Remark: Every proof which is based on RGM consists of three steps.

Step 1. The description of possible fixed points.

Step 2. The analysis of the spectra of linearized operators near fixed points.

Step 3. The description of the set of possible initial conditions, initial manifolds, etc.

Part II.

Several Results from the Mathematical Fluid Dynamics

In a big part of this text we consider the 3-dimensional Navier-Stokes system on R^3 for incompressible fluids with viscosity 1 without external forcing. In Part III we discuss the application of RGM to this system. It is written for the velocity vector $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ and for the pressure $p(x,t)$ and has the form:

$$\operatorname{div} u \,=\, 0$$

.

$$\frac{\mathcal{D}u}{dt} \,=\, \frac{\partial u}{\partial t} \,+\, (u, \nabla)\, u \,=\, \triangle u \,-\, \nabla p\,. \tag{5}$$

The first general results in the existence problem for (5) were proven by J. Leray (see [L]). Later important contributions were done by E. Hopf (see [H]), T. Kato (see [K]) and others.

An essential breakthrough appeared in the works of O. Ladyzenskaya (see her book [La]) where she proved the existence and uniqueness of strong solutions in the 2-dimensional case and bounded domains. Many mathematicians made important contributions to this field and I apologize for not being able to quote their results properly.

Our analysis of (5) begins with the Fourier transform of (5). It is quite natural from the point of the theory of dynamical systems because we are interested in solutions with singularities and it is not always easy to explain the meaning of spatial derivatives in the equation. This difficulty disappears if we make Fourier transform. The Fourier transform of (5) is written for C^3-functions $-iv(k,t)$ where $k \in R^3$ and $v(k,t) \perp k$ for every $k \neq 0$. The last property is equivalent to incompressibility in (5) and actually determines the phase space of dynamical system which we shall consider. Thus, instead of (5), we have

$$v(k,t) = \exp\{-t|k|^2\} \cdot v(k,0) + \int_0^t \exp\{-(t-s)|k|^2\}ds$$

$$\int_{R^3} < v(k-k',s),\, k > P_k\, v(k',s)\, d^3k' \,. \tag{6}$$

Here P_k is the orthogonal projection to the space orthogonal to k, i.e. $P_k\, v = v - \frac{<v,k>k}{<k,k>}$. The formula (6) gives a formal definition of the flow corresponding to the 3-dim Navier-Stokes system. In my opinion, (6) should become as popular as quadratic family of maps or geodesic flows and maybe more important.

There are many notable results related to the existence problem for (6). In this text, we shall describe only two of them.

Introduce the space $\Phi(\alpha)$ of functions $v(k) = \frac{c(k)}{|k|^\alpha}$ where $2 < \alpha < 3$ and $c(k) \perp k$ and is continuous everywhere except $k=0$, $\sup_{k \neq 0} |c(k)| < \infty$. This space is natural for the Navier-Stokes system. The power-like behavior near $k=0$ is connected with a power-like decay of solutions as $x \longrightarrow \infty$. Vice-versa, the decay at infinity is related to the smoothness of $u(x,t)$.

In the spaces $\Phi(\alpha)$ the local existence theorem is valid.

Theorem 3. *Let* $v(k,0) = \frac{c(k,0)}{|k|^\alpha}$ *and* $\sup\limits_{k \neq 0} |c(k,0)| = c^{(0)}$. *Then there exists an interval* $[0,T]$ *on the time axis and a function* $v(k,t) \in \Phi(\alpha)$ *defined for* $0 \le t \le T$ *which is continuous in* t *and satisfies (6). It is unique in* $\Phi(\alpha)$.

Theorems of this type are usually proven with the help of some iteration scheme. If $v(k,t) = \frac{c(k,t)}{|k|^\alpha}$ then the function $c(k,t)$ satisfies the equation

$$
\begin{aligned}
c(k,t) \,&=\, \exp\{-|k|^2 t\} \cdot c(k,0) + \\
&+ i|k|^\alpha \int_0^t \exp\{-|k|^2(t-s)\} \cdot ds \int\limits_{R^3} \frac{< k, c(k-k',s) > P_k c(k',s) dk'}{|k-k'|^\alpha \cdot |k'|^\alpha} \,.
\end{aligned}
$$

Put $c^{(0)}(k,t) = \exp\{-|k|^2 t\} \cdot c(k,0)$ and

$$
\begin{aligned}
c^{(n)}(k,t) \,&=\, \exp\{-|k|^2 t\} \cdot c(k,0) + \\
&+ i|k|^\alpha \int_0^t \exp\{-|k|^2(t-s)\}\, ds \int\limits_{R^3} \frac{< k, c^{(n-1)}(k-k',s) > P_k c^{(n-1)}(k',s)\, dk}{|k-k'|^\alpha \cdot |k'|^\alpha} \,.
\end{aligned}
$$

Assume that $\sup\limits_{\substack{k \in R^3 \\ 0 \le t \le T}} |c^{(n-1)}(k,t)| \,\le\, c^{(n-1)}$. Then

$$
|c^{(n)}| \le c^{(0)} + \, (c^{(n-1)})^2 \sup_{k \in R^3} |k|^{\alpha+1} \cdot \int_0^t \exp\{-|k|^2(t-s)\}\, ds \cdot \int\limits_{R^3} \frac{dk'}{|k-k'|^\alpha \cdot |k'|^\alpha} \,=
$$

$$
= \, c^{(0)} \,+\, (c^{(n-1)})^2 \cdot \sup_{k \in R^3 \setminus 0} |k|^{\alpha-1}(1 - \exp\{-|k|^2 t\}) \int\limits_{R^3} \frac{dk'}{|k-k'|^\alpha \cdot |k'|^\alpha} \,.
$$

For the last integral we have the estimate

$$\int\limits_{R^3} \frac{dk'}{|k-k'|^\alpha \cdot |k'|^\alpha} \leq \frac{B_1}{|k|^{2\alpha-3}}$$

where B_1 is a constant. Assume by induction that $c^{(n-1)} \leq 2c^{(0)}$. Then we have to show that

$$\sup_{k\in R^3\setminus 0} (2c^{(0)})^2 \, B_1 \, \frac{1}{|k|^{\alpha-2}} \cdot (1-\exp\{-|k|^2 t\}) \leq c^{(0)}$$

if t is small enough, or

$$\sup_{k\in R^3\setminus 0} \cdot 4c^{(0)} B_1 \frac{1}{|k|^{\alpha-2}} (1-\exp\{-|k|^2 t\}) \leq 1 \, .$$

This will give $c^{(n)} \leq 2c^{(0)}$.

Consider two cases.

<u>Case 1.</u> $|k|^2 \leq \frac{1}{t}$. Then

$$\frac{1}{|k|^{\alpha-2}} \cdot (1-\exp\{-|k|^2 t\}) \leq |k|^{4-\alpha} \cdot t \leq t^{\epsilon/2}$$

where $\epsilon = \alpha - 2$ and

$$\sup_{k\in R^3\setminus 0} 4c^{(0)} \frac{1}{|k|^{\alpha-2}} (1-\exp\{-|k|^2 t\}) \leq 4c^{(0)} t^\epsilon \leq 1$$

if $t \leq T$ and T is small enough.

Case 2. $|k|^2 \geq \frac{1}{t}$. Then

$$\frac{1}{|k|^{\alpha-2}} \cdot (1 - \exp\{-|k|^2 t\}) \leq \frac{1}{|k|^{\alpha-2}} \leq t^{\epsilon/2}$$

and again

$$\sup_{k \in R^3 \setminus 0} 4c^{(0)} B_1 \frac{1}{|k|^{\alpha-2}} (1 - \exp\{-|k|^2 t\}) \leq 4c^{(0)} \cdot t^{\epsilon/2} \leq 1$$

if T is small enough.

Thus all iterations $c^{(n)}(k,t)$, $0 \leq t \leq T$ have the norm less than $2c^{(0)}$.

The next step in the proof is to show that the iterations $c^{(n)}$ converge to a limit. We have

$$c^{(n)}(k,t) - c^{(n-1)}(k,t) = i|k|^\alpha \int_0^t \exp\{-|k|^2(t-s)$$

$$\left[\int_{R^3} \frac{< k, c^{(n-1)}(k-k',s) - c^{(n-2)}(k-k',s) > P_k c^{(n-1)}(k',s) dk'}{|k-k'|^\alpha \cdot |k'|^\alpha} \right.$$

$$\left. + \int_{R^3} \frac{< k, c^{(n-2)}(k-k',s) > P_k (c^{(n-1)}(k',s) - c^{(n-2)}(k',s)) dk'}{|k-k'|^\alpha \cdot |k'|^\alpha} \right]$$

and

$$|c^{(n)}(k,t) - c^{(n-1)}(k,t)| \leq 2c^{(0)} \parallel c^{(n-1)} \; c^{(n-2)} \parallel \cdot$$

$$\cdot |k|^{\alpha-1} \cdot (1 - \exp\{-|k|^2 t) \int_{R^3} \frac{dk'}{|k-k'|^\alpha \cdot |k'|^\alpha}$$

$$\leq\ 2c^{(0)}\ \cdot\ B_1 \cdot\ \parallel c^{(n-1)} - c^{(n-2)}\ \parallel\ \cdot\ \sup_{k\in R^3\smallsetminus 0} |k|^{\alpha-1}\ \cdot\ (1-\exp\{-|k|^2 t\})\ \cdot\ \frac{1}{|k|^{2\alpha-3}}\,.$$

The same arguments as before give that

$$\sup_{k\in R^3\smallsetminus 0} |k|^{\alpha}(1-\exp\{-|k|^2 t\})\ \cdot\ \frac{1}{|k|^{2\alpha-3}}\ \leq\ B_2 t^{\epsilon/2}$$

where B_2 is another absolute constant. Thus, for some constant B_3

$$\parallel c^{(n-1)} - c^{(n)} \parallel \leq\ B_3\ \cdot\ c^{(0)}\ \cdot\ t^{\epsilon/2} \parallel c^{(n-2)} - c^{(n-1)} \parallel\ .$$

This gives the needed convergence of the iterations and the uniqueness in $\Phi(\alpha)$ provided that T is small enough.

Take some $v(k,0)$. Assume that it can imbedded in different spaces $\Phi(\alpha)$. Then, it is easy to check that the solutions given for different α actually coincide.

LeJan and Sznitman proved in [LeJS] the following interesting theorem.

Theorem 4. *Assume that* $v(k,0) = \frac{c(k,0)}{|k|^2}$ *and* $\sup|c(k,0)| = c^{(0)} < \infty$, *i.e.,* $v(k,0) \in \Phi(2)$. *If* $c^{(0)}$ *is small enough then there exists a global solution of (6) defined for all* $t > 0$. *This solution is unique.*

Slightly different proofs of this theorem were given by Cannone and Planchon in [CP] and in [S2].

Both Theorems 3 and 4 cover many cases when a local and global existence theorems are valid. Presumably, the global existence theorem for small initial data is not valid in $\Phi(\alpha)$. However, M. Arnold and I proved in [AS] that for the analogous periodic problem it is true. Let me mention also the recent result by Dinaburg and myself (see [DS]) where we proved the local existence result and global existence result for small initial data in the case of initial conditions which are finite linear combinations of δ-functions.

Probably the spaces $\Phi(\alpha)$ is not optimal for our purposes. The space of functions $v(k)$ which have finite limits as $k \longrightarrow 0$ along any direction and this limit may depend on the direction are more suitable.

Proof of Theorem 4. Let

$$N\{c(k,t)\,, t \geq 0 \text{ and } k \neq 0\} \;=\; i|k|^2 \int_0^t \exp\{-|k|^2(t-s)\}\,ds \,\cdot$$

$$\cdot \int_{R^3} \frac{< k, c(k-k',s)\; > \; P_k\, c(k',s)\, dk'}{|k-k'|^2 \,\cdot\, |k'|^2}$$

It is easy to check that

$$I_2 \;=\; \int_{R^3} \frac{d^3k'}{|k-k'|^2 \,\cdot\, |k'|^2} \;\leq\; \frac{B_3}{|k|}\,.$$

where B_3 is an absolute constant. Then

$$|N\{c(k,t), t \,\geq\, 0 \text{ and } k \neq O\}| \;\leq\; C^2 \,\cdot\, B_3$$

where $C = \sup\limits_{k,t} |c(k,t)|$. This immediately implies the global existence result if C is small.

I believe that for $\alpha = 2$ and large initial conditions even the local existence result is not always true. The corresponding statement could be considered as some examples of blow up.

Part III.

Blow Ups of Complex Solutions of 3-dim Navier-Stokes System and RGM

We return back to the Fourier transform of the 3-dimensional Navier-Stokes System:

$$v(k,t) = \exp\{-t|k|^2\}\, v(k,0) \;+\; \int_0^t \exp\{-|k|^2(t-s)\}\, ds\,.$$

$$\int_{R^3} <\; v(k-k',s)\,,k \;>\; P_k\, v(k',s)\, d^3k' \tag{7}$$

$$v(k,t) \;\perp\; k \text{ for all } k \;\neq\; 0\,. \tag{8}$$

An important property of (7), (8) is the energy inequality: if $E(t) = \int_{R^3} < v(k,t), v(k,t) > d^3k$ and $-iv(k,t)$ is the Fourier transform of the real-valued $u(x,t)$ then

$$E(t') \;\leq\; E(t'') \text{ if } t' \;\geq\; t''\,. \tag{9}$$

However, (9) is no longer true if $v(k,t)$ are arbitrary C^3-functions i.e., solutions of (7), (8) are complex. In the text below we consider real solutions of (7) which in principle do not correspond to the real solutions in the x-space.

Power series for solutions of (7), (8). Denote $v_A(k,0) = A\, v(k,0)$ and $v_A(k,t)$ is the solution of (7), (8) with this initial condition, A is a real parameter. Write down the solution of (7) in the form

$$v_A(k,t) \;=\; \exp\{-|k|^2 t\}\, v_A(k,0) \;+\; \int_0^t \exp\{-|k|^2(t-s)\}ds \;\cdot\; \sum_{p>1} A^p\, h_p(k,s)\,. \tag{10}$$

The substitution of (10) into (7) gives the system of recurrent equations for the functions h_p:

$$h_1(k,s) \,=\, \exp\{-|k|^2 s\} \,\cdot\, v(k,0)\,,$$

$$h_2(k,s) \,=\, i\int\limits_{R^3} \,<\, v(k-k',0),\, k \,>\, P_k v(k',0)\, \exp\{-s|k-k'|^2 - s|k'|^2\} d^3k'\,,$$

$$h_p(k,s) = i\int\limits_0^s ds_2 \int\limits_{R^3} \,< v(k-k',\, 0, k \,>\, P_k h_{p-1}\,(k', s_2)\,,$$

$$\cdot\, \exp\{-s|k-k'|^2 - (s-s_2)|k'|^2\}\, d^3k' +$$

$$+\, i \sum_{\substack{p_1+p_2=p\,, \\ p_1,p_2>1}} \int\limits_0^s ds_1 \int\limits_0^s ds_2 \int\limits_{R^3} \,<\, h_{p_1}(k-k', s_1), k \,>\, \cdot\, P_k h_{p_2}(k', s_2)\,\cdot$$

$$\cdot\, \exp\{-(s-s_1)|k-k'|^2 - (s-s_2)|k'|^2\}\, d^3k' +$$

$$+\, i\int\limits_0^s ds_1 \int\limits_{R^3} \,< h_{p-1}\,(k-k', s_1), k \,>\, P_k\, v(k',0)\,\cdot$$

$$\exp\{-(s-s_1)|k-k'|^2 \,- s|k'|^2\}\, d^3k'\,. \tag{11}$$

This series was introduced in the papers [S1], [S2], [BDS]. It is possible to show that for bounded initial conditions with compact support and any given t it converges provided that A is sufficiently small.

The terms in (11) resemble the convolutions in probability theory. For example, if h_1 is concentrated in some subset $C \subset R^3$ then h_p are concentrated in $\underbrace{C + C + \ldots + C}_{p \text{ times}}$. The analogy with probability theory will be useful for our discussion below.

Below, I describe the main result of our joint paper with Dong Li (see [LS]).

Theorem. *There exists on open set in the space of 10-parameter families of initial conditions such that for each family from this set for some values of parameters one can find an interval on the time axis $S = [S^{(-)}, S^{(+)}]$ and a function $A(s), s \in S$, so that the solution with the initial condition $A(s)v(k,0)$ blows up at $t = s$.*

It will be explained below as to why we need 10 parameters and in what sense the solution blows up.

This result is obtained with the help of RGM. We derive the equation for the fixed point in our situation and show the existence of its solutions. Then we study the spectrum of the linearization. The number 10 is the sum of the number of unstable and neutral eigen-values.

Introduce the neighborhood

$$B_1 = \{k : |k - \kappa^{(0)}| \le \mathcal{D}_1 \sqrt{k^{(0)} \ln k^{(0)}}\}$$

where $\kappa^{(0)} = (0, 0, k^{(0)})$ and $k^{(0)}, \mathcal{D}_1$ are the main large parameters. Also

$$B_r = \{k : |k - r\,\kappa^{(0)}| \le \mathcal{D}_1 \sqrt{rk^{(0)} \ln(rk^{(0)})}\}\,.$$

Inside B_r introduce the rescaled coordinate Y using the formula $k = r\kappa^{(0)} + \sqrt{rk^{(0)}} \cdot Y$ and write down the representation for all $h_r\,, r < p$:

$$h_r(r\kappa^{(0)} + \sqrt{rk^{(0)}}\,Y, s) = Z_p(s) \cdot (\wedge_p(s))^r \cdot r \cdot g_r(Y, s)$$

where

$$g_r(Y,s) = \frac{1}{2\pi} \exp\left\{-\frac{Y_1^2 + Y_2^2}{2}\right\} \cdot \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{Y_3^2}{2}\right\} \cdot$$
$$\cdot (H_1(Y_1, Y_2) + \delta_1^{(r)}(Y_1, Y_2, Y_3), H_2(Y_1, Y_2) +$$
$$+ \delta_2^{(r)}(Y_1, Y_2, Y_3), \frac{1}{\sqrt{rk^{(0)}}} \cdot (F(Y_1, Y_2) + \delta_3^{(r)}(Y_1, Y_2, Y_3))\,.$$

Assume that $\delta_j^{(2)} \longrightarrow 0$ as $r \longrightarrow \infty$. Then we are in a situation similar to probability theory where H_1, H_2 play the role of Gaussian distribution. We shall derive the equation for

$H(Y_1, Y_2) = (H_1(Y_1, Y_2), H_2(Y_1, Y_2))$. The expression for F follows from the incompressibility condition:

$$H_1 Y_1 + H_2 Y_2 + F = O.$$

Introduce the variables θ_1, θ_2 instead of s_1, s_2 where $s_1 = s(1 - \frac{\theta_1}{(p_1 k^{(0)})^2})$, $s_2 = s(1 - \frac{\theta_2}{(p_2 k^{(0)})^2})$ and $\gamma = \frac{p_1}{p}$, $\kappa^{(0,0)} = (0, 0, 1)$. Then we can rewrite (11) as follows:

$$h_p(p\kappa^{(0)} + \sqrt{pk^{(0)}}Y, s) = Z_{p+1}(s) \wedge_{p+1}^{p}(s) \cdot p \cdot g_p(Y, s) = \ldots$$

$$+ i\, p \sum_{\substack{p_1+p_2=p \\ p_1,p_2>1}} \frac{1}{p} \frac{(pk^{(0)})^{5/2} \cdot p_1 \cdot p_2}{(p_1 k^{(0)})^2 (p_2 k^{(0)})^2} \int_0^{(p_1 k^{(0)})^2} d\theta_1 \int_0^{(p_2 k^{(0)})^2} d\theta_2$$

$$\int_{R^3} \left(\frac{1}{2\pi}\right)^{3/2} \exp\left\{- \frac{(Y_1 - Y_1')^2 + (Y_2 - Y_2')^2 + (Y_3 - Y_3')^2}{2\gamma}\right\}$$

$$\langle \tilde{g}_{p_1}\left(\frac{Y - Y'}{\sqrt{\gamma}}, s\left(1 - \frac{\theta_1}{(p_1 k^{(0)})^2}\right)\right), \kappa^{(0,0)} + \frac{Y}{\sqrt{pk^{(0)}}}\rangle \cdot$$

$$\cdot P_{\kappa^{(0,0)} + \frac{Y}{\sqrt{pk^{(0)}}}} \tilde{g}_{p_2}\left(\frac{Y'}{\sqrt{1-\gamma}}, s\left(1 - \frac{\theta_2}{(p_2 k^{(0)})^2}\right)\right)$$

$$Z_p\left(s\left(1 - \frac{\theta_1}{(p_1 k^{(0)})^2}\right)\right) \cdot \wedge_p^{p_1 - 1}\left(s\left(1 - \frac{\theta_1}{(p_1 k^{(0)})^2}\right)\right) \cdot$$

$$Z_p\left(s\left(1 - \frac{\theta_2}{(p_2 k^{(0)})^2}\right)\right) \wedge_p^{p_2 - 1}\left(s \cdot \left(1 - \frac{\theta_2}{(p_2 k^{(0)})^2}\right)\right) \cdot$$

$$\cdot \left(\frac{1}{2\pi}\right)^{3/2} \exp\left\{- \frac{(Y_1')^2 + (Y_2')^2 + (Y_3')^2}{2(1-\gamma)}\right\} \cdot \exp\left\{-\theta_1 |\kappa^{(0,0)} + \frac{Y - Y'}{\gamma\sqrt{pk^{(0)}}}|^2\right\}$$

$$\cdot \exp\left\{-\theta_2 |\kappa^{(0,0)} + \frac{Y'}{(1-\gamma)\sqrt{pk^{(0)}}}|^2\right\} + \cdots \qquad (12)$$

where dots mean the terms with $p_1 = 1$ and $p_1 = p-1$ which we do not write explicitly. We shall modify (12) neglecting by terms which tend to zero as $p \to \infty$. It is done in four steps.

Step 1. All terms $s\left(1 - \frac{\theta_1}{(p_1 k^{(0)})^2}\right)$ are replaced by s.

Step 2. Write

$$\frac{(pk^{(0)})^{5/2} \, \cdot \, p_1 p_2}{(p_1 k^{(0)})^2 \, \cdot \, (p_2 k^{(0)})^2} = \frac{(pk^{(0)})^{1/2}}{(k^{(0)})^2 \, \gamma(1-\gamma)} \, .$$

Step 3. Consider the inner product

$$(pk^{(0)})^{1/2} < g_{p_1} \left(\frac{Y - Y'}{\sqrt{\gamma}}, s \right) \, , \, \kappa^{(0,0)} \, + \, \frac{Y}{\sqrt{pk^{(0)}}} > \, .$$

Up to remainders it equals to

$$\left(\frac{1}{2\pi}\right)^{3/2} exp\left\{ - \frac{(Y_1 - Y_1')^2 \, + \, (Y_2 - Y_2')^2 \, + \, (Y_3 - Y_3')^2}{2\gamma} \right\} \cdot$$

$$\left[H_1 \left(\frac{Y_1 - Y_1'}{\sqrt{\gamma}} , \frac{Y_2 - Y_2'}{\sqrt{\gamma}} \right) Y_1 \, + \, H_2 \left(\frac{Y_1 - Y_1'}{\sqrt{\gamma}} , \frac{Y_2 - Y_2'}{\sqrt{\gamma}} \right) Y_2 \, + \, \frac{1}{\sqrt{\gamma}} \cdot \right.$$

$$\left. F\left(\frac{Y - Y'}{\sqrt{\gamma}} , s \right) \right] \, = \, \left(\frac{1}{2\pi}\right)^{3/2} \exp\left\{ - \frac{(Y_1 - Y_1')^2 \, + \, (Y_2 - Y_2')^2 \, + \, (Y_3 - Y_3')^2}{2\gamma} \right\} \cdot$$

$$\left\{ - \frac{\gamma - 1}{\sqrt{\gamma}} \left[H_1 \left(\frac{Y_1 - Y_1'}{\sqrt{\gamma}} , \frac{Y_2 - Y_2'}{\sqrt{\gamma}} \right) \frac{Y_1 - Y_1'}{\sqrt{\gamma}} + \right. \right.$$

$$\left. + H_2 \left(\frac{Y_1 - Y_1'}{\sqrt{\gamma}} , \frac{Y_2 - Y_2'}{\sqrt{\gamma}} \right) \frac{Y_2 - Y_2'}{\sqrt{\gamma}} \right] +$$

$$+ \sqrt{1-\gamma} \left[H_1 \left(\frac{Y_1 - Y_1'}{\sqrt{\gamma}} , \frac{Y_2 - Y_2'}{\sqrt{\gamma}} \right) \frac{Y_1'}{\sqrt{1-\gamma}} + \right.$$

$$\left. \left. + H_2 \left(\frac{Y_1 - Y_1'}{\sqrt{\gamma}} , \frac{Y_2 - Y_2'}{\sqrt{\gamma}} \right) \frac{Y_2'}{\sqrt{1-\gamma}} \right] \right\} \, .$$

Step 4. Replace the projection operator by the identity operator. It is reasonable because for $Y = O(1)$, $k_3 \sim pk^{(0)}$ the projection in the main order of magnitude is the projection to the plane (Y_1, Y_2).

The sum over p_1 is the Riemannian integral sum for the corresponding integral. Take $Z_p(s) = (k^{(0)})^2$ and write $\frac{\wedge_{p+1}(s)}{\wedge_p(s)} = 1 + \frac{\xi_{p+1}}{p^2}$. Then

$$\exp\left\{-\frac{|Y|^2}{2}\right\} \cdot \frac{1}{2\pi} H(Y) = \int_0^1 d\gamma \int_{R^2} \frac{1}{2\pi\gamma} \exp\left\{-\frac{|Y-Y'|^2}{2\gamma}\right\}$$

$$\frac{1}{2\pi(1-\gamma)} \cdot \exp\left\{-\frac{1}{2(1-\gamma)} |Y'|^2\right\} \cdot$$

$$\cdot \left[-(1-\gamma)^{3/2} \cdot \left(\frac{Y_1 - Y_1'}{\sqrt{\gamma}} H_1\left(\frac{Y-Y'}{\sqrt{\gamma}}\right) + \frac{Y_2 - Y_2'}{\sqrt{\gamma}} H_2\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)\right)\right.$$

$$\left. + \sqrt{\gamma}(1-\gamma)\left(\frac{Y_1'}{\sqrt{1-\gamma}} H_1\left(\frac{Y-Y'}{\sqrt{\gamma}}\right) + \frac{Y_2'}{\sqrt{1-\gamma}} H_2\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)\right)\right] \cdot H\left(\frac{Y'}{\sqrt{1-\gamma}}\right) d^2Y' . \tag{13}$$

This is our basic equation for the fixed point in RGM. Actually, the full equation has two additional parameters which we did not include here (see [LS]).

The analysis of this equation can be done if we use the expansions over Hermite polynomials. In [LS] the following theorem was proven.

Theorem 5. *The equation (13) has a three-parameter family of formal solutions. If these parameters are small, then the solutions are non-formal in the sense that they are given by bounded functions of Y.*

The parameters are not independent in the sense that some of them generate the same solutions. If we use all parameters then our family of fixed points depends on four parameters.

The next step in RGM is to introduce the "tangent space," the group of linearized maps and to study its spectrum. In our setting, the tangent space is simple and consists of functions $\delta(\gamma, Y)$, $0 \le \gamma \le 1$, $Y \in R^3$. We assume that for each $\gamma, 0 \le \gamma \le 1$, the function $\delta(\gamma, Y)$ belongs to the Hilbert space $L^2 = L^2(R^3)$ of square-integrable functions with respect to the weight $\left(\frac{1}{2\pi}\right)^{3/2} \exp\left\{-\frac{1}{2}|Y|^2\right\}$, $Y = (Y_1, Y_2, Y_3)$ and as a function of γ it is a continuous curve in this Hilbert space, $\max\limits_{0\le\gamma\le 1} \parallel \delta(\gamma, Y) \parallel_{L^2} < \infty$.

The semi-group $\{A^t\}$ of linearized maps acts as follows: if $\gamma \exp\{t\} < 1$ then $A^t\delta(\gamma, Y) = \delta(\gamma \exp t, Y)$. At $\gamma = 1$ we impose boundary conditions:

$$\exp\left\{-\frac{Y_1^2 + Y_2^2 + Y_3^2}{2}\right\} \cdot \frac{1}{(2\pi)^{3/2}}\, \delta(1, Y) = \int\limits_0^1 d\gamma \int\limits_{R^3} \frac{1}{(2\pi\gamma)^{3/2}} \cdot \frac{1}{(2\pi(1-\gamma))^{3/2}} \cdot$$

$$\exp\left\{-\frac{|Y_1 - Y_1'|^2 + |Y_2 - Y_2'|^2 + |Y_3 - Y_3'|^2}{2} - \frac{|Y_1'|^2 + |Y_2'|^2 + |Y_3'|^2}{2(1-\gamma)}\right\} \cdot$$

$$\cdot \left\{\left[-(1-\gamma)^{3/2} \left(\frac{Y_1 - Y_1'}{\sqrt{\gamma}}\, H_1\left(\frac{Y - Y'}{\sqrt{\gamma}}\right) + \frac{Y_2 - Y_2'}{\sqrt{\gamma}}\, H_2\left(\frac{Y - Y'}{\sqrt{\gamma}}\right)\right)\right.\right.$$

$$+\gamma^{1/2}\,(1-\gamma) \left(\frac{Y_1'}{\sqrt{1-\gamma}}\, H_1\left(\frac{Y - Y'}{\sqrt{\gamma}}\right) + \frac{Y_2'}{\sqrt{1-\gamma}}\, H_2\left(\frac{Y - Y'}{\sqrt{\gamma}}\right)\right)$$

$$\delta\left(1-\gamma\,,\, \frac{Y'}{\sqrt{1-\gamma}}\right) + \left[-(1-\gamma)^{3/2} \left(\frac{Y_1 - Y_1'}{\sqrt{\gamma}}\, \delta_1\left(\gamma,\, \frac{Y - Y'}{\sqrt{\gamma}}\right)\right.\right.$$

$$\left.\left. + \frac{Y_2 - Y_2'}{\sqrt{\gamma}}\, \delta_2\left(\gamma,\, \frac{Y - Y'}{\sqrt{\gamma}}\right)\right) + \gamma^{\frac{1}{2}}(1-\gamma) \left(\frac{Y_1'}{\sqrt{1-\gamma}}\, \delta_1\left(\gamma\,,\, \frac{Y - Y'}{\sqrt{\gamma}}\right) + \frac{Y_2'}{\sqrt{1-\gamma}} \cdot\right.\right.$$

$$\left.\left.\left.\left. \cdot\, \delta_2\left(\gamma\,,\, \frac{Y - Y'}{\sqrt{\gamma}}\right)\right)\right)\right]\, H\left(\frac{Y'}{\sqrt{1-\gamma}}\right)\, d^3Y'\,.\right.$$

It is not too difficult to show that the eigen-functions of the linearized group have the form:

$$\delta(\gamma, Y) \,=\, \gamma^{\alpha}\, \Phi_{\alpha}(Y)$$

where the functions Φ_α satisfy the equation which is similar to the equation for the fixed point. Again, its analysis can be done with the help of expansion into series with respect to Hermite polynomial and the result is the following (see [LS]):

Theorem 6. *The spectrum of the semi-group of the linearized maps consists of the following numbers:*

$$spec\,\{L^t\} \,=\, \left\{1, \frac{1}{2}, 0;\, \lambda_m^{(1)},\, \lambda_m^{(2)},\, m \geq 1\right\}$$

where the multiplicity of 1 is 1, the multiplicity of $\frac{1}{2}$ is 3, the multiplicity of zero is 6.

The eigen-values $\lambda_m^{(1)} = -\frac{m}{2}$, $\lambda_m^{(2)} = \frac{\sqrt{17}-4-m}{2}$, $m \geq 1$ and their multiplicities are $\nu_{\lambda_m^{(1)}} = \frac{(m+3)(m+4)}{2}$, $\nu_{\lambda_m^{(2)}} = \frac{m(m+5)}{2}$.

The eigen-values $\alpha = 1$, $\alpha = \frac{1}{2}$ are unstable and the eigen-value $\alpha = 0$ is neutral. Their multiplicities are 4 and 6 respectively. In view of the general ideology of RGM (see the beginning of these notes), 10-parameter families of initial condition which are C'-close to $H^{(nu)}$ intersect the stable manifold $\Gamma^{(s)}$. For points from $\Gamma^{(s)}$ the remainders $\delta^{(p)}(Y) \longrightarrow 0$ as $p \longrightarrow \infty$.

More detailed description of the open set of 10-parameter families of initial conditions.

Take $k^{(0)}$ which will be assumed to be sufficiently large and consider the neighborhood

$$A_1 \,=\, \left\{k : |k - k^{(0)}| \,\leq\, \mathcal{D}_1 \sqrt{k^{(0)}\, \ln k^{(0)}}\right\}$$

where $\mathcal{D}_1$ is also sufficiently large. Our initial conditions are zero outside A_1. Inside A_1 they have the form:

$$v(k,0) \;=\; \tfrac{1}{2\pi}\,\exp\left\{-\tfrac{Y_1^2+Y_2^2}{2}\right\}\,(H^{(0)}(Y_1,Y_2)+$$

$$+\sum_{j=1}^{4} b_j^{(u)}\,\Phi_j^{(u)}\,(Y_1,Y_2,Y_3)\;+\;\sum_{j'=1}^{b} b_{j'}^{(n)}\,\Phi_{j'}^{(n)}(Y_1,Y_2,Y_2)$$

$$+\,\Phi(Y_1,Y_2,Y_3,b^{(u)},b^{(n)}))\,\tfrac{1}{\sqrt{2\pi}}\,\exp\left\{-\tfrac{Y_3^2}{2}\right\}.$$

In this expression $k=k^{(0)}+\sqrt{k^{(0)}}Y$, $H^{(0)}(Y_1,Y_2)=(H_1^{(0)}(Y_1,Y_2),\,H_2^{(0)}(Y_1,Y_2),0)$ is the fixed point of RGM. $\Phi_{j'}^{(n)}$, $\Phi_{j'}^{(n)}$ are unstable and neutral eigen-functions of the semi-group of the linearized maps near the fixed point, $b_j^{(u)}$ and $b_{j'}^{(n)}$ are our main parameters, $-\rho_1 \le b_j^{(u)}$, $b_{j'}^{(n)} \le \rho_1$ where ρ_1 is another constant, $\Phi(Y_1,Y_2,Y_3;b^{(u)},b^{(n)}$ is in some sense small. Due to the presence of $b^{(u)}$, $b^{(n)}$ we have 10-parameters families of initial conditions, due to the presence of Φ we have an open set in the space of such families.

Having these initial conditions, we write the reprsentation of functions h_r in the form

$$h_r(k,s)\;=\;Z(s)\cdot\wedge^r(s)\;\cdot\;r\;\cdot\;\exp\left\{-\frac{Y^2}{2}\right\}\cdot\left(\frac{1}{\sqrt{2\pi}}\right)^3\cdot(H(Y)\;+\;\delta_r(Y,s))$$

if $|Y|\;\le\;\mathcal{D}_1\sqrt{\ln(rk^{(0)})}$ and $k\;=\;r\kappa^{(0)}\;+\;\sqrt{rk^{(0)}}\;\cdot\;Y$, $\kappa^{(0)}\;=\;(0,0,k^{(0)})$, $H(Y)$ is one of our fixed points. Outside the domain $|Y|\le\mathcal{D}_1\sqrt{\ln(rk^{(0)})}$ the function h_r satisfies simple power-like estimate. Parameter s changes within a fixed interval $s\in[S_-,S_+]$ of positive length. We adjust the parameters $b^{(u)}$, $b^{(n)}$ in such a way that the remainders δ_r tend to zero as $r\to\infty$. Take $A(s)=\wedge^{-1}(s)$. Then the solution with the initial datum $A(s)v(k,0)$ blows up at time s. The function $A(s)$ is increasing, i.e., the function $\wedge(s)$ is decreasing. This is natural because blow up at small s requires large initial conditions.

If $s'<s$ then $E(s')$, $\Omega(s')$ are finite. It is possible to show that $E(s')\asymp\frac{1}{(s-s')^5}$, $\Omega(s')\asymp\frac{1}{(s-s')^7}$. The series which gives $v_{A(s)}(k,s)$ generates linear functional on the space of bounded functions with compact support. It is not clear whether our solution can be extended beyond the point of singularity.

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Part II.

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