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MOM TECHNOLOGY AND VOLUMES OF HYPERBOLIC 3-MANIFOLDS

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0. Introduction

This paper is the first in a series whose goal is to understand the structure of low-volume complete orientable hyperbolic 3-manifolds. Here we introduce *Mom technology* and enumerate the hyperbolic Mom-n manifolds for $n \leq 4$. Our long-term goal is to show that all low-volume closed and cusped hyperbolic 3-manifolds are obtained by filling a hyperbolic Mom-n manifold, $n \leq 4$ and to enumerate the low-volume manifolds obtained by filling such a Mom-n.

William Thurston has long promoted the idea that volume is a good measure of the complexity of a hyperbolic 3-manifold (see, for example, [Th1] page 6.48). Among known low-volume manifolds, Jeff Weeks ([We]) and independently Sergei Matveev and Anatoly Fomenko ([MF]) have observed that there is a close connection between the volume of closed hyperbolic 3-manifolds and combinatorial complexity. One goal of this project is to explain this phenomenon, which is summarized by the following:

Hyperbolic Complexity Conjecture 0.1. *(Thurston, Weeks, Matveev-Fomenko) The complete low-volume hyperbolic 3-manifolds can be obtained by filling cusped hyperbolic 3-manifolds of small topological complexity.*

Remark 0.2. Part of the challenge of this conjecture is to clarify the undefined adjectives *low* and *small*. In the late 1970's, Troels Jorgensen proved that for any positive constant C there is a finite collection of cusped hyperbolic 3-manifolds from which all complete hyperbolic 3-manifolds of volume less than or equal to C can be obtained by Dehn filling. Our long-term goal stated above would constitute a concrete and satisfying realization of Jorgensen's Theorem for "low" values of C.

A special case of the Hyperbolic Complexity Conjecture is the long-standing

Smallest Hyperbolic Manifold Conjecture 0.3. The Weeks Manifold M_W, obtained by $(5,1)$, $(5,2)$ surgery on the two components of the Whitehead Link, is the unique oriented hyperbolic 3-manifold of minimum volume.

Note that the volume of M_W is $0.942\ldots$.

All manifolds in this paper will be orientable and all hyperbolic structures are complete. We call a compact manifold *hyperbolic* if its interior supports a complete hyperbolic structure of finite volume.

Definition 0.4. A *Mom-n structure* (M, T, Δ) consists of a compact 3-manifold M whose boundary is a union of tori, a preferred boundary component T, and a handle decomposition Δ of the following type. Starting from $T \times I$, n 1-handles

[1]Partially supported by NSF grants DMS-0071852, DMS-0346270, and DMS-0504110.
Version 0.80, July 27, 2006.

and n 2-handles are attached to $T \times 1$ such that each 2-handle goes over exactly three 1-handles, counted with multiplicity. Furthermore, each 1-handle encounters at least two 2-handles, counted with multiplicity. We say that M is a *Mom-n* if it possesses a Mom-n structure (M, T, Δ).

Remarks 0.5. On a Mom-n, the handle decomposition Δ deformation retracts to an almost simple 2-complex which has $2n$ true vertices, in the sense of Matveev [Mv2]. Therefore Mom-n manifolds are a subset of those with Matveev complexity at most $2n$.

Here is the fundamental idea at the foundation of our project. Given a complete finite-volume hyperbolic 3-manifold N, start with either a slightly shrunken maximal horotorus neighborhood V of a cusp or slightly shrunken maximal tube V about a geodesic. After expanding V in the normal direction, it eventually encounters itself, thereby creating a 1-handle. Subsequent expansions give rise to the creation of 1, 2, and 3-handles. In the presence of low volume we expect that V will rapidly encounter 1 and 2-handles and ∂V together with a subset of these handles (perhaps somewhat perturbed to allow for the "valence-3 2-handle condition") will create a Mom-n manifold M, for some $n \leq 4$. Furthermore, the complement of M will consist of cusp neighborhoods and tubular neighborhoods of geodesics. In practice, the handle structure may arise in a somewhat different manner; e.g., as a sub-complex of the dual triangulation of the Ford Domain (see [GMM3]).

The papers [GM] and [GMM] can be viewed as steps in this direction when V is a tubular neighborhood about a geodesic γ. Indeed, [GM] gives a lower bound on $\mathrm{Vol}(N)$ in terms of the tube radius of γ and [GMM] gives a lower bound in terms of the first two ortholengths, or equivalently the radii of the expanding V as it encounters its first and second 1-handles.

Definition 0.6. If $i : M \to N$ is an embedding, then we say that the embedding is *elementary* if $i_*\pi_1(M)$ is abelian, and *non-elementary* otherwise.

In §1 we give the basic definitions regarding Mom-n manifolds embedded in hyperbolic 3-manifolds and state for later use some standard results about hyperbolic 3-manifolds and embedded tori in such. The end result of §2 - §4 is that if $n \leq 4$, then given a non-elementary Mom-n in a hyperbolic manifold one can find a non-elementary hyperbolic Mom-k, where $k \leq n$. Weaker results are given for general values of n. In §5 we enumerate the hyperbolic Mom-n manifolds for $n \leq 4$; Theorem 5.1 and Conjecture 5.2 together imply the following:

Theorem 0.7. *There are* 3 *hyperbolic Mom-*2 *manifolds, 21 hyperbolic Mom-*3 *manifolds (including the 3 hyperbolic Mom-*2*'s, which are also Mom-*3*'s).*

Conjecture 0.8. *There are 138 hyperbolic Mom-*4 *manifolds (including the hyperbolic Mom-*2*'s and Mom-*3*'s, which are also Mom-*4*'s).*

In §6 we show that any non-elementary embedding of a hyperbolic Mom-n manifold M, $n \leq 4$, into a compact hyperbolic manifold N gives rise to an *internal Mom-n structure* on N, i.e. every component of ∂M either splits off a cusp of N or bounds a solid torus in N.

In §7 we give examples of internal Mom-2 structures on cusped hyperbolic 3-manifolds, including in particular a detailed exposition of one of our key motivating examples.

In future papers we will use the Mom Technology to directly address the Hyperbolic Complexity Conjecture, and the Smallest Hyperbolic Manifold Conjecture. Indeed, in [GMM3] we will identify all 1-cusped hyperbolic 3-manifolds with volume less than 2.7 by showing that all such manifolds possess internal Mom-n structures with $n \leq 3$. This result, in combination with work of Agol-Dunfield (see [AST]), gives a lower bound of 0.86 for the volume of an orientable hyperbolic 3-manifold (but see the note below). The Agol-Dunfield result is an improvement of an earlier result of Agol which provides a tool for controlling the volume of a hyperbolic 3-manifold in terms of the volume of an appropriate cusped hyperbolic 3-manifold from which it is obtained via Dehn filling; the improved version utilizes recent work of Perelman.

This leads to three very promising directions towards the Smallest Hyperbolic Manifold Conjecture. Either, improve the technology of [GMM3] to identify the 1-cusped manifolds of volume less than 2.852, and then apply Agol-Dunfield. Or, extend the tube radius results of [GMT] from $\log(3)/2$ to 0.566 (the Agol-Dunfield volume bound involves the radius of a solid tube around a short geodesic in the original closed hyperbolic 3-manifold). Or, extend the method of [GMM3] to the closed case, thereby providing an essentially self-contained proof of the Smallest Hyperbolic Manifold Conjecture.

Note that all three approaches require an analysis of volumes of hyperbolic 3-manifolds obtained by Dehn filling a Mom-2 or Mom-3 manifold. These Dehn filling spaces have been extensively studied by J. Weeks and others, and it is highly likely that all low-volume manifolds in these Dehn filling spaces have been identified. However, some work will need to be done to bring these studies up to a suitable level of rigor.

The authors wish to thank Morwen Thistlethwaite for his assistance in identifying the hyperbolic manifolds in §5.

1. Basic Definitions and Lemmas

Definition 1.1. Let M be a compact connected 3-manifold M with $B \subset \partial M$ a compact surface which may be either disconnected or empty. A *handle structure* Δ on (M, B) is the structure obtained by starting with $B \times I$ adding a finite union of 0-handles, then attaching finitely many 1 and 2-handles to $B \times 1$ and the 0-handles. We call $B \times I$ (resp. $B \times I \cup$ 0-handles) the *base* (resp. *extended base*) and say that the handle structure is based on B. The *valence* of a 1-handle is the number of times, counted with multiplicity, the various 2-handles run over it and the *valence* of a 2-handle is the number of 1-handles, counted with multiplicity, it runs over.

Following the terminology of Schubert [Sch] and Matveev [Mv1] we call the 0-handles, 1-handles and 2-handles *balls*, *beams* and *plates* respectively. We call *islands* (resp. *bridges*) the intersection of the extended base with beams (resp. plates) and the closed complement of the islands and bridges in $B \times 1 \cup \partial$(0-handles) are the *lakes*. We say that Δ is *full* if each lake is a disc. If $B = \emptyset$, then we say that Δ is a *classical handle structure*.

Let M be a compact 3-manifold with ∂M a union of tori and let T be a component of ∂M. We say that (M, T, Δ) is a *weak Mom-n* if Δ is a handle structure based on T without 0-handles or 3-handles and has an equal number of 1 and 2-handles, such that each 1-handle is of valence ≥ 2 and each 2-handle is of valence

2 or 3. Furthermore, there are exactly n 2-handles of valence 3. A weak Mom-n with no valence-2 2-handles is a *Mom-n*. A weak Mom-n is *strictly weak* if there exists a valence-2 2-handle.

The following is a well-known existence result stated in our language.

Proposition 1.2. *A compact 3-manifold M has a weak Mom-structure if and only if ∂M is a union of at least two tori.*

Proof. If M has a weak Mom-n structure, then by definition all of its boundary components are tori and there is at least one such boundary component. Further, because there are no 3-handles in Δ, there must be another (torus) boundary component.

The converse is not much more difficult. In fact, if M has at least two boundary components, then it is standard to create it by starting with a thickened boundary component and adding 1 and 2-handles where the 1 and 2-handles are of valence ≥ 2. By subdividing the 2-handles with 1-handles we satisfy the condition that the 2-handles have valence ≤ 3. Since $\chi(M) = 0$, there are an equal number of 1 and 2-handles. □

Definition 1.3. Call a torus that bounds a solid torus a *tube* and call a torus bounding a tube with knotted hole a *convolutube*. Recall that a *tube with knotted hole* is a $B^3 - \overset{\circ}{N}(\gamma)$, where γ is a knotted proper arc.

The following standard result follows from the loop theorem (see for example [Jac]).

Lemma 1.4. *If S is a torus in an irreducible 3-manifold N, then either S is incompressible or S is a tube or a convolutube. If $S \subset \overset{\circ}{N}$, ∂N is incompressible and S is compressible, and there exists an embedded essential annulus connecting S to a component of ∂N, then S is a tube.* □

Proposition 1.5. *If M is a non-elementary compact, connected 3-manifold embedded in the compact hyperbolic 3-manifold N and ∂M is a union of tori, then, up to isotopy, N is obtained from M by first filling a subset of the components of ∂M by solid tori to obtain the manifold M_1, then compressing a subset of the components of ∂M_1 to obtain the manifold M_2, then attaching 3-balls to the 2-sphere components of ∂M_2 to obtain M_3. Furthermore all of these operations can be performed within N.*

Proof. The components of ∂M that bound solid tori in N are exactly those boundary components which compress to the non-M side. Fill in all such tori to obtain the manifold M_1. If P is a component of ∂M_1 which is not boundary parallel in N, then P is compressible in N and hence is a convolutube. These convolutubes can be isotoped to lie in pairwise disjoint 3-balls in N. Therefore we can compress all the compressible components of ∂M_1 (to obtain M_2) and cap the resulting 2-spheres with 3-cells to obtain M_3 which is isotopic to N.

Since M_3 must have all boundary components boundary parallel in N and M_3 is non-elementary, the result follows. □

Corollary 1.6. *Let $M \subset \overset{\circ}{N}$ be a connected compact non-elementary submanifold in the compact hyperbolic 3-manifold N. If ∂M is a union of tori, then each component of ∂N is parallel to a component of ∂M via a parallelism disjoint from M.* □

The following result is due to Kerckhoff (see [Koj]).

Lemma 1.7. *If γ is a simple closed geodesic in the complete, finite-volume hyperbolic 3-manifold N, then $N - \gamma$ has a complete finite-volume hyperbolic structure.* □

Lemma 1.8. *Let M be a compact submanifold of the compact hyperbolic 3-manifold N.*

i) If $M_1 = M - \mathring{X}$, where $X \subset N$ is either a solid torus or 3-ball with $\partial X \subset M$, or M_1 is obtained by deleting a 2-handle or more generally deleting an open regular neighborhood of a properly embedded arc from M, then the inclusion $M \to N$ is a non-elementary embedding if and only if the inclusion $M_1 \to N$ is a non-elementary embedding.

ii) Suppose M is non-elementary, ∂M is a union of tori and A is an essential annulus in M. Split M along A; then some component M_1 of the resulting manifold is non-elementary.

iii) Suppose $F \subset M$ is an embedded torus essential in M, and $M \subset N$ is non-elementary. Split M along F; then exactly one component of the resulting manifold is non-elementary.

Proof. The conclusion is immediate in case (i) because both M_1 and M have the same π_1-image.

Under the hypotheses of (ii) the boundary of each component of M split along A is also a union of tori. We consider the case where the split manifold connected, for the general case is similar. Since all tori in N separate, M is obtained from M_1 by attaching a thickened annulus A to a boundary parallel torus, a tube or a convolutube. In the first case M and M_1 have the same π_1-image since N is anannular. If A is attached to the outside of a convolutube, then $\pi_1(M) \subset \mathbb{Z}$. If A is attached to the outside of a tube, then M can be enlarged to a Seifert fibered space in N and hence is elementary.

To prove (iii) note that F is either boundary parallel or is a tube or convolutube. In each case the result follows immediately. □

Definition 1.9. Let N be a compact hyperbolic 3-manifold. An *internal Mom-n structure on N* consists of a non-elementary embedding $f : M \to N$ where (M, T, Δ) is a Mom-n and each component of ∂M is either boundary parallel in N or bounds a solid torus in N. We will sometimes suppress mention of the embedding and simply say that (M, T, Δ) is an internal Mom-n structure on N. In the natural way we define the notion of *weak internal Mom-n structure on N.*

Lemma 1.10. *A non-elementary embedding of the Mom-n manifold M into the compact hyperbolic 3-manifold N will fail to give an internal Mom-n structure on N if and only if some component of ∂M maps to a convolutube. In that case, a reimbedding of M, supported in a neighborhood of the convolutubes gives rise to an internal Mom-n structure on N.* □

Definition 1.11. A *general based Mom-n* (M, B, Δ) consists of a compact manifold M with ∂M a union of tori, $B \subset \partial M$ a compact codimension-0 submanifold of ∂M that is π_1-injective in ∂M, and Δ a handle structure based on B without 0-handles such that every 1-handle is of valence-≥ 2, every 2-handle is of valence-3 and there are exactly n of each of them. A *weak general based Mom-n* is as above with Δ

perhaps having $k \geq 0$ extra valence-2 2-handles in addition to the n valence-3 2-handles and hence has $k+n$ 1-handles.

A *general based internal Mom-n structure on N* consists of a non-elementary embedding $f: M \to N$, where N is a compact hyperbolic 3-manifold and (M, B, Δ) is a general based Mom-n structure. Along similar lines we have the notion of *weak general based internal Mom-n structure on N.*

Notation 1.12. If X is a space, then let $|X|$ denote the number of components of X, $\overset{\circ}{X}$ denote the interior of X and $N(X)$ denote a regular neighborhood of X. If σ is a 2-handle, then let $\delta\sigma$ denote the lateral boundary, i.e. the closure of that part of $\partial\sigma$ which does not lie in lower index handles. If b is a bridge which lies in the 2-handle σ, then define $\delta b = b \cap \delta\sigma$.

2. Handle structures and Normal surfaces

We slightly modify Haken's [Ha] theory of surfaces in handlebodies to our setting. We closely parallel the excellent exposition given by Matveev in [Mv1].

Definition 2.1. Let Δ be a handle structure on M based on $B \subset \partial M$. A compact surface $F \subset M$ is called *normal* if

(1) F intersects each plate $D^2 \times I$ in parallel copies of the form $D^2\times$ pt.
(2) Each component of the intersection of F with a beam $D^2 \times I$ is of the form $\alpha \times I$, where α is a proper arc whose endpoints are disjoint from δ(bridges). Furthermore, each component of $D^2 \times 0 - \alpha$ intersects δ(bridges) in at least two points.
(3) Each component U of $F \cap B \times [0,1] \cup$ 0-handles is π_1-injective in $B \times [0,1] \cup$ 0-handles. If $U \cap B \times 0 \neq \emptyset$, then U is a product disc or annulus, i.e. The inclusion $(U, U \cap (B \times 0), U \cap (B \times 1), U \cap (\partial B \times I)) \to (B \times I, B \times 0, B \times 1, \partial B \times I)$, can be relatively isotoped to a vertical embedding.
(4) If U is a component of $F \cap B \times I$ with $F \cap B \times 0 \neq \emptyset$, then U is an essential vertical disc or annulus, i.e. $U = \alpha \times I$ where $\alpha \times 0$ is either an essential simple closed curve or essential proper arc in $B \times 0$.

Remark 2.2. i) For F closed, the second condition can be restated by requiring that α intersect distinct components of $D^2 \times 0 \cap$ (bridges). When $\partial F \neq \emptyset$, the second condition implies that F is locally efficient in that it neither can be locally boundary compressed nor can its weight be reduced via an isotopy supported in the union of a 2-handle and its neighborhing 1-handles.

ii) Note that ∂F lies in the union of the beams, lakes and $B \times 0$.

Lemma 2.3. *(Haken) If F is a compact, incompressible, boundary-incompressible surface in a compact irreducible 3-manifold, then F is isotopic to a normal surface.* □

Definition 2.4. Let Δ be a handle structure on M based on $B \subset \partial M$. The *valence* $v(b)$ of a beam (resp. plate) is the number of plates (resp. beams) that attach to it, counted with multiplicity. Define the *complexity* $C(\Delta)$ to be $(\rho_1(\Delta), |\Delta^1|)$ lexicographically ordered, where $\rho_1(\Delta) = \Sigma_{\text{beams}\, b} \max(v(b) - 2, 0)$ and $|\Delta^1|$ is the number of 1-handles. In particular we have

ρ_1-**formula**: $\rho_1(\Delta) = \sum_{\text{2-handles } \sigma} \text{valence}(\sigma) - 2|\Delta^1| + |\text{valence-1 1-handles}| + 2|\text{valence-0 1-handles}|$

Lemma 2.5. *(Matveev) Let Δ be a handle structure on M based on B, $F \subset M$ a closed normal surface and let M' be M split along F. If each component of $M' \cap B \times [0,1]$ disjoint from $B \times 0$ is a 3-ball, then M' has a handle structure Δ' based on B with $\rho_1(\Delta') = \rho_1(\Delta)$.*

Proof. This follows almost exactly as in §3 and §4 of [Mv1]: M' naturally inherits a handle structure Δ_1 from Δ as follows. The surface F splits $B \times I$ into various submanifolds one of which is homeomorphic to $B \times [0,1]$ with $B \times 0 = B$. All of the other submanifolds which lie in M' are 3-balls. This new $B \times [0,1]$ becomes the base and the 3-balls become 0-handles. The various 1 and 2-handles are split by F into 1 and 2-handles and as in [Mv1], $\rho_1(\Delta_1) = \rho_1(\Delta)$. □

Lemma 2.6. *Given the handle structure Δ on (M, B), if some 1-handle is valence-1, then there exists another structure Δ_1 on (M, B) with $C(\Delta_1) < C(\Delta)$.* □

Lemma 2.7. *If (M, T, Δ) is a Mom-n, then $C(\Delta) = (n, n)$.* □

Lemma 2.8. *Let Δ be a handle structure on (M, B), $F \subset M$ a connected separating normal surface and M_1 be the component of $M - \mathring{N}(F)$ which does not contain B. If each component of $F \cap B \times [0,1]$ is a disc, then M_1 has a classical handle structure Δ_1 with $\rho_1(\Delta_1) \leq \rho_1(\Delta)$.*

Proof. This follows as in the proof of Lemma 2.5 after noting that each component of $M_1 \cap B \times [0,1]$ is a 3-ball and these 3-balls correspond to the 0-handles of the induced handle structure on M_1. □

Lemma 2.9. *If ∂M is a union of tori, and Δ is a handle structure on (M, T) with T a component of ∂M, then there exists a weak Mom-n (M, T, Δ_1) with $n \leq \rho_1(\Delta_1)$.*

Proof. First apply Lemma 2.6, then add 1-handles to subdivide the valence-k, $k \geq 4$, 2-handles into valence-3 2-handles. □

Definition 2.10. If $B \neq \emptyset$ is a compact submanifold of ∂M, then define $\mathrm{rank}_{\rho_1}(M, B)$ to be the least n such that there exists a handle decomposition Δ on (M, B) with $\rho_1(\Delta) = n$.

Problem 2.11. *Is there an example of a compact hyperbolic 3-manifold N with T a component of ∂N and A an essential annulus in T such that $\mathrm{rank}_{\rho_1}(N, A) < \mathrm{rank}_{\rho_1}(N, T)$?*

3. Estimates for the reduction of ρ_1 under splitting

The main result of these next two sections is Theorem 4.1 which shows that if a hyperbolic 3-manifold N has an internal Mom-n structure (M, T, Δ) with Δ full and $n \leq 4$, then it has an internal Mom-k structure (M_1, T_1, Δ_1) where $k \leq n$, Δ_1 is full, and M_1 is hyperbolic. If $n > 4$, we obtain the similar conclusion except that "full" is replaced by "general based" and hence T_1 can be a union of tori and annuli.

As far as we know, transforming a structure based on an annulus lying in T_1 to one based on the whole torus T_1 may require an increase in ρ_1. This issue is responsible for many of the technicalities of this section and the next. See Problem 2.11.

Lemma 3.1. *Let $f : M \to N$ be a non-elementary embedding of a compact connected manifold into a compact irreducible 3-manifold. Suppose ∂M is a union of tori, T is a component of ∂M and that Δ is a handlebody structure on (M, T) without 0-handles such that each 2-handle is of valence ≥ 3. Then there exists a non-elementary embedding $g : M' \to N$ with $\partial M'$ a union of tori and a handle structure Δ' on (M', T') with $T' = T$ such that*

$$\rho_1(\Delta') + 2|\textit{valence-1 1-handles of } \Delta| + 3|\textit{valence-0 1-handles of } \Delta| \leq \rho_1(\Delta).$$

If instead each 2-handle of Δ is of valence ≥ 2 then we have

$$\rho_1(\Delta') + |\textit{valence-1 1-handles of } \Delta| + 2|\textit{valence-0 1-handles of } \Delta| \leq \rho_1(\Delta).$$

Proof. Both assertions follow similarly by induction on the number of 1-handles of Δ. If η is a valence-1 1-handle, then cancelling η with its corresponding 2-handle creates a handle structure Δ_1. The Lemma follows by applying the ρ_1-formula to Δ_1 and induction. If η is a valence-0 1-handle, then the manifold M_1 obtained by deleting η is connected and $\chi(M_1) = 1$, hence has a 2-sphere boundary component S which bounds a 3-ball disjoint from $\overset{\circ}{M}_1$. Let (M_2, T, Δ_2) be obtained from (M, T, Δ) by deleting η as well as a 2-handle which faces S. M_2 is a non-elementary embedding by Lemma 1.8. Now apply the ρ_1-formula and induction. □

Lemma 3.2. *Let $f : M \to N$ be a non-elementary embedding of a manifold into a compact hyperbolic 3-manifold N, where ∂M is a union of tori. Suppose that M has a full handle structure Δ without 0-handles based on the component T of ∂M such that every 2-handle is of valence ≥ 3. If either of the following are true then there exists a non-elementary embedding $f : M' \to N$ with handle structure Δ' on (M', T') such that $\rho_1(\Delta') + 2 \leq \rho_1(\Delta)$, $T' = T$ and $\partial M'$ is a union of tori:*

i) There exists a valence-1 2-handle $\sigma \subset N - \overset{\circ}{M}$ that can be added to Δ.

ii) There exists a disc $D \subset \partial M$ such that ∂D is the union of two arcs $\alpha \cup \beta$, where β lies in a lake and α lies in a 2-handle λ. Furthermore, within $\lambda \cap \partial M$, α separates components of $\lambda \cap (1 - \text{handles})$.

Proof. By Lemma 3.1 we can assume that every 1-handle of Δ is of valence ≥ 2. To prove (i) let Δ_1 be the handle structure on the manifold M_1 obtained by attaching σ to Δ along ∂M. Let η denote the 1-handle which σ meets. Let Δ_2 and M_2 be obtained by deleting a 2-handle $\lambda \neq \sigma$, which faces the resulting 2-sphere boundary component. Let Δ_3 be obtained by cancelling σ and η. Finally, in the usual way, reduce to a non-elementary M_4 with structure Δ_4 on (M_4, T) whose 1 and 2-handles are of valence ≥ 2. Applying the ρ_1-formula shows that $\rho_1(\Delta_4) + 2 \leq \rho_1(\Delta)$ unless, measured in Δ, valence(η) = valence$(\lambda) = 3$ and λ attaches to η at least twice. If λ attaches to η twice, then in the passage from Δ_1 to Δ_2 delete a 2-handle $\lambda_1 \neq \lambda$ which faces the 2-sphere. If λ attaches to η thrice, then either Δ is not full or η is the unique 1-handle of Δ. In the latter case $M_2 = T \times I$ which is elementary, a contradiction.

ii) Under these hypotheses we can attach a 2-handle $\sigma \subset N - \overset{\circ}{M}$ to Δ such that either valence(σ)=1 or valence$(\sigma) \leq$ valence$(\lambda) + 2$ and λ faces the resulting 2-sphere boundary component. If valence$(\sigma) = 1$, then apply (i). Otherwise let Δ_1 be obtained by deleting λ and apply the ρ_1-formula and if necessary Lemma 3.1. □

Lemma 3.3. *Let M be a non-elementary embedding of a compact 3-manifold into the compact hyperbolic 3-manifold N with ∂M a union of tori, and let Δ be a handle structure of M based on $R \subset \partial M$. If there exists a valence ≥ 3 1-handle η of Δ which attaches to a 0-handle ζ, then there exists a non-elementary embedding $M' \to N$, and a handle structure Δ' based on $R' \subset \partial M'$ such that $\rho_1(\Delta') < \rho_1(\Delta)$. Here either $(M', R') = (M, R)$ or $M' = M - \overset{\circ}{V}$ and $R' = R \cup \partial V$ where V is an embedded solid torus in $\overset{\circ}{M}$.*

Proof. If η also attaches to either the base or a 0-handle distinct from ζ, then cancelling η with ζ gives rise to Δ' on (M, R) with $\rho_1(\Delta') < \rho_1(\Delta)$. If η attaches only to ζ, then let M' be obtained by deleting $\overset{\circ}{V}$, the open solid torus gotten by hollowing out ζ and η. Let $R' = R \cup \partial V$ and let Δ' be the induced structure on (M', R'). □

Lemma 3.4. *Let N be a compact hyperbolic 3-manifold. If (M, T, Δ) is a full internal weak Mom-n structure on N and M is reducible then there exists (M_p, T_p, Δ_p), a weak internal Mom-k structure on N such that $k + 2 = \rho_1(\Delta_p) + 2 \leq \rho_1(\Delta) = n$. If T does not lie in a 3-cell, then $T_p = T$.*

Proof. We first consider the case that no reducing sphere in M bounds a ball in N containing T. Let F be a least-weight normal reducing 2-sphere and note that $F \cap T \times I$ is a union of discs. Let M_0 and M_0' be the components of M split along F. By Lemma 1.8 and the irreducibility of N exactly one of M_0, M_0' is non-elementary. We let M_0 (resp. M_0') denote the non-elementary (resp. elementary) component with Δ_0 (resp. Δ_0') its induced structure. By hypothesis $T \subset M_0$.

We show that $\rho_1(\Delta_0') > 0$. Let X denote the union of the islands and bridges of Δ and $Y' = X \cap M_0'$. If $\rho_1(\Delta_0') = 0$, then each component A of Y' is an annulus. If some component A of Y' is disjoint from the lakes of Δ, then Y' would be disjoint from the lakes and so F would 2-fold cover a projective plane, which contradicts the fact that N is irreducible. Therefore each component of Y' has one boundary component in a lake and one component in $\overset{\circ}{X}$. Since Δ is full, this implies that F is a boundary parallel 2-sphere and contradicts the fact that ∂M is a union of tori.

If $Y = X \cap M_0$ then a similar argument shows that some component of Y is not an annulus and furthermore some 2-handle σ of Δ_0 faces F and attaches to a valence-$\geq$ 3 1-handle. Delete σ from Δ_0 to obtain (M_1, T, Δ_1) with $\rho_1(\Delta_1) + 1 \leq \rho_1(\Delta_0)$. The standard simplifying moves as in Lemma 3.1 transform (M_1, T, Δ_1) to (M_2, T, Δ_2), a weak Mom-k with $k = \rho_1(\Delta_2) \leq \rho_1(\Delta_1)$. Since $\rho_1(\Delta_0) + \rho_1(\Delta_0') = \rho_1(\Delta) = n$ we have $k + 2 = \rho_1(\Delta_p) + 2 \leq \rho_1(\Delta) = n$. If M_2 is reducible, then split along an essential least-weight sphere F_2 which is normal with respect to Δ_2. Retain the component which contains T and do the usual operations to obtain the weak Mom-k_2 (M_3, T, Δ_3) with $k_2 \leq k$ and M_3 non-elementary. By Haken finiteness, this procedure terminates in a finite number of steps, completing the proof in this case.

We now consider the case that T is compressible in M. Let F be a least-weight compressing disc for T. Note that $F \cap T \times I$ consists of discs and a single annulus. If M_1 is M split along F with induced handle structure Δ_1 and A is T split along $F \cap T$, then Δ_1 is based on A. Since M is obtained by attaching a 1-handle to a spherical component of ∂M_1, it follows that M_1 is non-elementarily embedded in N. Let M_2 be obtained by attaching 2-handles to ∂M_1 along the components of ∂A, then capping off the resulting 2-sphere which faces A with a 3-cell. The resulting manifold M_2 has two 2-sphere boundary components and the induced handle structure Δ_2 is classical. Since Δ is full some 2-handle of Δ_2 facing a 2-sphere of ∂M_2 attaches to a 1-handle of valence ≥ 3. Delete this 2-handle to obtain the non-elementary M_3 with handle structure Δ_3 which satisfies $\rho_1(\Delta_3)+1 \leq \rho_1(\Delta)$. Delete another 2-handle to create M_4 and Δ_4 such that ∂M_4 is a union of tori. Now apply Lemma 3.3 to Δ_4 to create (M_5, T_5, Δ_5) so that $\rho_1(\Delta_5)+1 \leq \rho_1(\Delta_4)$. Finally cancel the extraneous 0-handles and valence-1 1-handles to create the desired weak internal Mom-k structure (M_6, T_6, Δ_6).

Let F be a least-weight essential normal 2-sphere for Δ. Since we can assume that T is incompressible in M, $F \cap T \times I$ is a union of discs. If T does not lie in the ball bounded by $F \subset N$, then proceed as in the first part of the proof. Otherwise let M_0 and M_0' be the components of M split along F, with M_0' the component containing T. Since Δ is full and some component of $T \times 1 \cap M_0'$ is nonplanar, it follows that $\rho_1(\Delta_0') \geq 1$ and hence $\rho_1(\Delta_0) + 1 \leq \rho_1(\Delta)$. To complete the proof apply Lemma 3.3 to Δ_0, delete a 2-handle to create a manifold with torus boundary components and cancel low valence handles as in the previous paragraph. □

Remark 3.5. If (M, T, Δ) is a weak internal Mom-n structure on the compact hyperbolic manifold N, and ∂M is compressible in M, then M is reducible. Therefore, if M is irreducible and Δ has an annular lake A that is homotopically inessential in $T \times 1$, then the core of A bounds a disc in $T \times I$ which separates off a 3-cell in M. Absorbing this 3-cell into $T \times I$ simplifies Δ and transforms A into a disc lake.

From now on we will assume that if a homotopically inessential lake appears it is immediately removed via the above operation.

Definition 3.6. If Δ is a handle structure on M, then the *sheets* of Δ are the connected components of the space $\mathcal{S}$ which is the union of the 2-handles and the valence-2 1-handles. So sheets are naturally thickened surfaces which are attached to a 3-manifold along their thickened boundaries. The valence of a sheet is the number of times the boundary runs over 1-handles counted with multiplicity.

Lemma 3.7. *Let N be a compact hyperbolic 3-manifold and $f : M \to N$ a non-elementary embedding where ∂M is a union of tori. Let Δ be a handle structure on M with no 0-handles based on a component T of ∂M such that the valence of each 2-handle is at least 3 and T does not lie in a 3-cell of N. If some sheet S of Δ is not a thickened disc then there exists a non-elementary $M' \subset N$ with handle structure Δ' based on a component T' of $\partial M'$ such that $\rho_1(\Delta') + 1 \leq \rho_1(\Delta)$. If equality must hold then S is a thickened annulus or Mobius band and if Δ is full then S is a thickened Mobius band and Δ' is full.*

Proof. By Lemma 3.1 and the proof of Lemma 3.4 we can assume that M is irreducible and each 1-handle of Δ has valence ≥ 2. If M_1 denotes the manifold obtained by deleting the sheet S, then $\chi(M) = \chi(M_1) + \chi(S)$. Since $\chi(M) = 0$, if $\chi(S) < 0$, then ∂M_1 contains a 2-sphere. This implies that M is reducible. Note that if $\chi(S) = 0$, then M_1 is non-elementary.

Now assume that $\chi(S) = 0$. In this case S is either an annulus$\times I$ or a non-trivial I-bundle over a Mobius band. Since $\chi(S) = 0$, if S contains more than one 2-handle, then valence$(S) \geq 2$. If valence$(S) > 1$, then $\rho_1(\Delta_1) + 2 \leq \rho_1(\Delta)$ holds where Δ_1 is the induced structure on M_1. If S is a valence-1 annulus$\times I$, then Δ has an annular lake and $\rho_1(\Delta_1)+1=\rho_1(\Delta)$.

If S is a thickened Mobius band of valence-1, then $\rho_1(\Delta_1)+1=\rho_1(\Delta)$. If S attached to the component R of ∂M_1 and R had an annular lake, then R must be a tube with a compressing disc D whose boundary goes over a 1-handle η of Δ_1 exactly once. Let M_2 be obtained by attaching the 2-handle σ with core D to M_1 and Δ_2 the induced handle structure. Proceed as in the proof of Lemma 3.2 to show that exists a non-elementary embedding (M_3, T) with handle structure Δ_3 with $\rho_1(\Delta_3)+2\le\rho_1(\Delta_1)+1\le\rho_1(\Delta)$. □

Lemma 3.8. *Let (M,T,Δ) be an internal Mom-n structure on the compact hyperbolic manifold N. Assume that every sheet of Δ is a disc and Δ is full. If there exists an embedded annulus A connecting the component S of $\partial M - T$ to T, then there exists a non-elementary embedding of a manifold M' into N such that $\partial M'$ is a union of tori and $\rho_1(\Delta')+2\le\rho_1(\Delta)$. Here Δ' is a handle structure on M' based on T'. Either $T'=T$ or T' is the union of an essential annulus in T and possibly a component of $\partial M'$.*

Proof. By Lemma 3.4 we can assume that M is irreducible. Since all sheets are discs and letting $M_2=M$, we obtain from Δ a full handle structure Δ_2 with no 0-handles based on the component $T_2=T$ of ∂M_2 where each 1 and 2-handle of Δ_2 is of valence ≥ 3. Finally $\rho_1(\Delta_2)\le\rho_1(\Delta)$. We can assume that $A\subset M_2$ is a least-weight normal annulus connecting T_2 to the boundary component $S\ne T_2$. Since A is least weight, $A\cap T_2\times I$ is a union of discs and a single annulus.

Let ∂A_0 (resp. ∂A_1) denote $A\cap T_2$ (resp. $A\cap S$). Our A has an induced handle structure Φ based on ∂A_0 as follows. The base consists of the annular component of $A\cap T_2\times[0,1]$, the 0-handles consist of the disc components of $A\cap T_2\times[0,1]$, the 1-handles (resp. 2-handles) consist of the intersections of A with the 1-handles (resp. 2-handles).

Let M_3 denote the manifold obtained by splitting M_2 along A and let Δ_3 denote the induced handle structure. As in §2, the ball components of $T_2\times I$ split along A to become 0-handles of Δ_3. The remaining component is $B'\times I$, where B' is T_2 split along ∂A_0 and hence Δ_3 is based on $B'\subset T_2$. By Lemma 1.8, M_3 is non-elementarily embedded in N.

If η is a 1-handle of Δ_2 and if $\{\eta_i\}$ denotes the 1-handles of Δ_3 which descended from η, then $\sum_i(\text{valence}(\eta_i)-2)\le\text{valence}(\eta)-2$ with equality if and only if A_1 does not run over η. In fact, counting with multiplicity, if A_1 runs over the 1-handles of Δ_2 more than once then by operations as in Lemma 3.1 we can pass to M_4 and Δ_4 with $\rho_1(\Delta_4)+2\le\rho_1(\Delta)$. We will now assume that A_1 ran over a unique 1-handle η and it did so with multiplicity one. This implies that $\rho_1(\Delta_3)+1=\rho_1(\Delta_2)$. Also A_1 is the union of two arcs α and β, where α lies in a 1-handle of Δ_2 and β lies in a lake. β lies either in the base of Φ or in a 0-handle v^* of Φ.

Give A a transverse orientation. Call a 0-handle v of Φ *plus* (resp. *minus*) if the transverse orientation of the disc $v\subset T_2\times I$ points away from (resp. towards) T_2. Each such disc v separates off, in $T_2\times I$, a 3-ball v_B. Let v_D denote $v_B\cap T_2\times 1$.

If β lies in a 0-handle v^*, then Lemma 3.3 applies to Δ_3 and the Lemma is proved. Indeed, since A is least weight, the disc v^*_D contains a bridge b in its interior and this bridge is not parallel to β. In other words, there is no embedded disc in a lake whose boundary is a concatenation of four arcs β_1,β_2,β_3 and β_4 such that $\beta_1=\beta,\beta_3$ lies in ∂b, and β_2 and β_4 are arcs lying in islands. Otherwise, since A_1

runs over a unique 1-handle, this implies that Δ_2 had a valence-1 2-handle. If v_D^* is disjoint from the other 0-handles of Φ, then Lemma 3.3 applies to the 0-handle v_B^* of Δ_3. Otherwise, v_B^* is split into balls by the various 0-handles of Φ and Lemma 3.3 applies to one of these balls.

From now on we assume that β lies in the base of Φ. Let $X \subset T_2 \times 1$ denote the union of the islands and bridges of Δ_2. A similar but easier argument to the one given in the previous paragraph shows that for each 0-handle v of Φ either ∂v is boundary parallel in X or Lemma 3.3 applies. This implies that if $v \neq w$ are 0-handles of Φ, and $v_B \subset w_B$, then ∂v and ∂w are normally parallel in X. Furthermore, no 1-handle of Φ connects a plus 0-handle to a minus 0-handle of Φ. Also, if v^0 and v^1 are two 0-handles of Φ that are connected by a 1-handle, then $v_B^0 \cap v_B^1 = \emptyset$. Finally, there do not exist 0-handles $w^0, w^1, \cdots w^n$ of Φ such that for $i = 1, 2, \cdots, n-1$, w^i is connected to w^{i+1} by a 1-handle and $w_B^n \subset w_B^0$.

It follows that there exists a disc $E \subset A$ whose boundary is the union of two arcs ϕ and ψ where ϕ is a proper arc in a 2-handle of Φ and ψ lies in the $T \times 1 \cap (\text{base}(A))$. Furthermore $E \cap (0, 1\text{-handles } \Phi) \neq \emptyset$ and is connected. By the previous paragraph if v, w are 0-handles of Φ lying in E, then they are of the same parity and $v_B \cap w_B = \emptyset$.

Therefore, E can be normally isotoped, with respect to Δ_2 to a disc $G \subset \partial M$ such that ∂G is a union of two arcs, one lying in a lake and the other in a 2-handle. Now apply (ii) of Lemma 3.2 to Δ_2. □

Lemma 3.9. *Let (M, T, Δ) be a full internal Mom-n structure on the compact hyperbolic 3-manifold N such that every sheet is a disc. Suppose that there exists an essential embedded annulus A with $\partial A \cap T = \emptyset$. Then either there exists a full internal Mom-k structure (M', T', Δ') on N with $k < n$ or there exists a non-elementary embedding $M' \to N$ with handle structure (M', T', Δ') where $\partial M'$ is a union of tori, T' is a component of $\partial M'$, and $\rho_1(\Delta') + 2 \leq \rho_1(\Delta)$.*

Proof. Assume that M is irreducible. By Lemma 3.8 we can assume that no essential annulus connects T to a component of $\partial M - T$. Let A be a least-weight essential normal annulus with $A \cap T = \emptyset$. By the second sentence, $A \cap T \times I$ is a union of discs. Let M_1 be the non-elementary component of M split along A and let (M_1, T_1, Δ_1) be the induced handle structure. As in the previous proof, $\rho_1(\Delta_1) + 1 \leq \rho_1(\Delta)$. Indeed, since ∂A traverses at least two 1-handles of Δ, counted with multiplicity, the inequality will be strict unless A normally double covers a Mobius band. If M_1 is disjoint from T, then the result follows from Lemma 3.3. Assume now that $T \subset \partial M_1$ and Δ_1 is full.

Cancel the 0-handles with 1-handles to obtain (M_2, T_2, Δ_2). If Δ_2 is not full, then some 1-handle of valence ≥ 3 was cancelled and hence ρ_1 is reduced. Next cancel valence-1 2-handles to obtain (M_3, T_3, Δ_3). Again fullness is preserved or ρ_1 is reduced. If Δ_3 has a valence-2 2-handle σ connecting distinct 1-handles, then cancel one of these 1-handles with σ. If the 2-handle σ attaches to the same 1-handle α, then the core of $\sigma \cap \alpha$ can be viewed as an embedded annulus or Mobius band C in M_3. Splitting M_3 along C we obtain (M_4, T_4, Δ_4) where T_4 is the newly created boundary component of M_4 and Δ_4 is the induced structure. Note that $\rho_1(\Delta_4) < \rho_1(\Delta_3)$ since Δ_3 is full. After finitely many such operations the lemma is proved. □

Lemma 3.10. *Let (M, T, Δ) be a full internal Mom-n structure on the compact hyperbolic 3-manifold N such that every sheet is a disc. Suppose that there exists an* *essential embedded annulus A with $\partial A \subset T$. Then either there exists a full internal Mom-k structure (M', T', Δ') on N with $k < n$ or there exists a non-elementary embedding $M' \to N$ with handle structure (M', T', Δ') where $\partial M'$ is a union of tori, T' is a component of $\partial M'$, and $\rho_1(\Delta') + 2 \leq \rho_1(\Delta)$.*

Proof. If M is reducible or there exists an essential annulus with some boundary component disjoint from T, then apply Lemma 3.4, Lemma 3.8 or Lemma 3.9. Let $(M, \partial M - T, \Sigma)$ be the dual handle structure. I.e. the 1-handles (resp. 2-handles) of Σ are in 1-1 correspondence with the 2-handles (resp. 1-handles) of Δ and for each handle σ, the core of σ is the cocore of the dual handle and vice versa. Note that $\rho_1(\Sigma) = \rho_1(\Delta)$ and each 1-handle of Σ is of valence 3, but Σ may have valence-2 2-handles.

Let A be an essential annulus, least weight with respect to Σ. Since each essential annulus has its entire boundary in T, $A \cap (\partial M - T) \times I$ is a union of discs. Let M_1 be the non-elementary component of M split along A with (M_1, R_1, Σ_1) the induced structure. As before $\rho_1(\Sigma_1) < \rho_1(\Sigma)$. Eliminate the 0-handles and low valence-1 and 2-handles of Σ_1 to obtain (M_2, R_2, Σ_2) and note that either Σ_2 is full or $\rho_1(\Sigma_2) + 2 \leq \rho_1(\Sigma) = \rho_1(\Delta)$. If Σ_2 is full, then (M_3, T_3, Δ_3) the handle structure dual to (M_2, R_2, Σ_2) is full. Let (M_4, T_4, Δ_4) be obtained by eliminating the valence-2 2-handles as in the previous proof. As before $\rho_1(\Delta_4) + 1 \leq \rho_1(\Delta)$ with equality holding only if Δ_4 is full. □

Lemma 3.11. *Let (M, T, Δ) be a full internal Mom-n structure on the compact hyperbolic 3-manifold N. If M is not hyperbolic, then either there exists a full internal Mom-k structure (M', T', Δ') on N with $k < n$ and M' hyperbolic or there exists a general based internal Mom-l, $l + 2 \leq n$, structure (M', B', Δ') on N.*

Proof. By Lemma 3.4, 3.7, 3.8, 3.9 or 3.10 it suffices to consider the case that M is irreducible and anannular. By Thurston [Th2], if M is not hyperbolic, then it contains an essential torus. Let F be a least-weight essential torus. Since there are no essential annuli, $F \cap T \times I$ is a union of discs.

Let M_1 denote the component of M split along F which contains T and let M_1' denote the other component. Let Δ_1 and Δ_1' denote the induced handle structures. Note that Δ_1 is based on T, Δ_1' is a classical structure and $\rho_1(\Delta_1) + \rho_1(\Delta_1') = \rho_1(\Delta)$. By Lemma 1.8 one of M_1 or M_1' is non-elementary . Let $X \subset T \times 1$ be the union of the islands and bridges of Δ, $Y = M_1' \cap X$ and $Z = M_1 \cap X$.

If M_1 is elementary, $\rho_1(\Delta_1) \geq 1$ since Z is nonplanar. In the usual way obtain a handle structure Σ on M_1' such that $\rho_1(\Sigma) \leq \rho_1(\Delta')$, Σ has exactly one 0-handle and each 1-handle is of valence ≥ 2. Since M_1' is non-elementary, some 1-handle is of valence ≥ 3. To complete the proof apply Lemma 3.3.

From now on assume that M_1 is non-elementary. We first show that $\rho_1(\Delta_1') \geq 1$. If $\rho_1(\Delta_1') = 0$, then Y is a union of annuli, each component of which intersects ∂X in ≤ 1 circle, since Δ is full. If $Y \cap \partial X \neq \emptyset$, then each component must have this property and hence F is boundary parallel in M. If $Y \cap \partial X = \emptyset$, then M_1' is an I-bundle over a Klein bottle, which is impossible in an orientable hyperbolic 3-manifold.

In the usual way pass from (M_1, T_1, Δ_1) to a Mom structure (M_2, T_2, Δ_2) with $\rho_1(\Delta_2) \leq \rho_1(\Delta_1)$. Since M is anannular, Δ_2 has no annular lakes. If some component of ∂M_2 is a convolutube, then reimbed M_2 in N to get a full internal Mom-k structure where $k < n$. □

4. From Mom-n to Hyperbolic Mom-n

The following is the main result of this section:

Theorem 4.1. *If (M, T, Δ) is a full internal Mom-n structure on the compact hyperbolic 3-manifold N and $n \leq 4$, then there exists a full internal Mom-k structure (M', T', Δ') on N where M' is hyperbolic and $k \leq n$.*

For general n, either there exists an internal Mom-k structure (M', T', Δ') in N with $k \leq n$ and M hyperbolic or there exists a general based internal Mom-k structure (M', B', Δ') on N such that M' is hyperbolic and each component of B' is either a component of $\partial M'$ or an annulus which is essential in a component of ∂M. If $M \neq M'$, then $k + 2 \leq n$.

We first prove some preliminary lemmas about complexity (see Definition 2.4).

Lemma 4.2. *(Clean-Up Lemma) Let M be a compact 3-manifold with ∂M a union of tori and N a compact hyperbolic 3-manifold. If $M \to N$ is a non-elementary embedding and Δ is a handle structure on (M, B), then there exists (M', B', Δ') a weak general based internal Mom-n structure on N with $C(\Delta') \leq C(\Delta)$.*

If M is hyperbolic, then either $\rho_1(\Delta') < \rho_1(\Delta)$ or $M' = M$.

Proof. Proof by induction on $C(\Delta)$. By the usual cancellation operations pass to a handle structure Δ_2 on (M, B) without 0-handles or 1 and 2-handles of valence 1. If Δ has a valence-0 1-handle, then delete it and proceed as in Lemma 3.1 to obtain Δ_3 on (M_3, B_3) with M_3 non-elementary and ∂M_3 a union of tori. All of these operations reduce $C(\Delta)$.

If Δ_2 has a valence-0 2-handle σ, then compress M_3 along a disc D which passes once through σ and intersects $B \times I$ in an annulus. Let Δ_4 be the resulting handle structure on (M_4, B_4) where M_4 is the non-elementary component of M_3 split along D and B_4 is $B_3 \cap M_4$. This splits σ into two 2-handle components, at least one of which lies in Δ_4. Absorb such 2-handles into $B_4 \times I$. This creates (M_5, B_5, Δ_5) where $M_5 = M_4$ and B_5 is obtained by attaching one or two discs to ∂B_4. If some component of B_5 is a 2-sphere S, then create (M_6, B_6, Δ_6) by filling in S with a 3-cell E and identifying $S \times I \cup E$ as a 0-handle of Δ_6. Now cancel with a 1-handle to obtain (M_7, B_7, Δ_7) with $C(\Delta_7) < C(\Delta_3)$. If one or two components of B_5 are discs and there exists a non-simply connected component of B_5, then the ball components of $B_5 \times I$ are now re-identified as 0-handles and cancelled with 1-handles to obtain (M_7, B_7, Δ_7). If all the components of B_5 are discs, then transforming the components of $B_5 \times I$ to 0-handles produces a classical handle structure $\Delta_{5.5}$ on M_5. Now apply the proof of Lemma 3.3 to create (M_6, B_6, Δ_6) with $C(\Delta_6)$ reduced. If some component of ∂M_6 is a 2-sphere, then delete a 2-handle facing it to obtain (M_7, B_7, Δ_7).

By repeatedly applying the above operations we can assume that our Δ_7 has no 0-handles, and each 1-handle and 2-handle is of valence ≥ 2. Furthermore, no component of B_7 is a disc. If some component G of B_7 is not π_1-injective in ∂M_7 then attach a 2-handle to a component of $\partial G \times I$ whose restriction to ∂M_7 bounds a disc, to create (M_8, B_8, Δ_8). Delete a 2-handle which faces the resulting 2-sphere boundary component and simplify as in the previous paragraphs to create (M_9, B_9, Δ_9) with $C(\Delta_9) < C(\Delta_7)$. We can now assume that each component of B_9 is essential in ∂M_9, Δ_9 is 0-handle free and all 1 and 2-handles are valence ≥ 2.

If ∂M_9 contains convolutubes, then reimbed M_9 in N to eliminate them. If M is not hyperbolic, then the resulting $(M_{10}, B_{10}, \Delta_{10})$ satisfies the conclusions of our clean up lemma.

If M is hyperbolic, then the proof also follows as above. However, note that ∂M is incompressible so in the above process valence-0 1-handles or lakes compressible in $B \times I$ can be eliminated without changing M and with reducing complexity. The topology of M may change if we delete a 2-handle or apply Lemma 3.3; however, if such operations must be done, they can be done in such a manner that reduces ρ_1. □

Lemma 4.3. *If (M, B, Δ) is a strictly weak general based Mom-n structure with M hyperbolic, then there exists a general based Mom-k structure (M, B', Δ') with $C(\Delta') < C(\Delta)$.*

Proof. Proof by induction on $C(\Delta)$. If σ is a valence-2 2-handle which goes over distinct 1-handles, then cancelling a 1-handle with σ creates (M, B, Δ_1) with $C(\Delta_1) < C(\Delta)$.

We now assume that no valence-2 2-handle σ goes over distinct 1-handles. Suppose that σ goes over the same 1-handle α twice. Then $\sigma \cup \alpha$ can be viewed as an embedded annulus or Mobius band A with boundary on B. Since M is hyperbolic and orientable, A is either a boundary parallel annulus or a compressible annulus.

If A is boundary parallel, then it together with an annulus on T cobound a solid torus $V \subset B$ such that A wraps longitudinally around V exactly once. Melting V into $B \times [0, 1]$ eliminates both α and σ, together with all the 2-handles and 1-handles inside of V. The resulting manifold (M, B_1, Δ_1) is a weak provisional Mom-k with $k \leq n$ whose induced handle structure Δ_1 satisfies $C(\Delta_1) < C(\Delta)$.

If A is not boundary parallel, then M is reducible, a contradiction. □

Lemma 4.4. *If (M, B, Δ) is a general based Mom-n and M is hyperbolic, then $n \geq 2$.*

Proof. This follows by direct calculation. □

Lemma 4.5. *If the compact hyperbolic 3-manifold N has a general based internal Mom-2 structure (M, B, Δ), then it has a full internal Mom-2 structure.*

Proof. By splitting along annuli in $B \times I$ we can assume that every non-peripheral lake of B is a disc. If B is the union of two tori T_1 and T_2, then each 1-handle must connect T_1 to T_2. This implies that each 2-handle is of even valence which is a contradiction. If B consists of a single torus, then (M, B, Δ) is a full Mom-2 structure. We finally assume that B contains an annulus. We only discuss the case that B is connected; the case where B is either the union of an annulus and torus or two annuli is similar and easier.

First suppose that Δ has a single sheet of valence four. Let η denote the valence-4 1-handle of Δ. There exists an essential compressing disc D for $B \times I$ which cuts across the bridges in at most three components. View $N(D)$ as a 1-handle and $B \times I - \mathring{N}(D)$ as a 0-handle to obtain a classical handle structure with two 1-handles respectively of valence 4 and ≤ 3. Now as in Lemma 3.3 hollow out the 0-handle and η to get a non-elementary M_1 with handle structure Δ_1 based on a torus with a single 1-handle of valence ≤ 3. Therefore we obtain a Mom-1 structure on a hyperbolic 3-manifold, which is a contradiction.

Now consider the case that Δ has two valence-3 1-handles. If some essential disc D in $B \times I$ cuts the bridges in ≤ 2 components, then as above, we obtain a handle structure with one 0-handle and three 1-handles. Hollowing out a valence-3 1-handle and the 0-handle produces a handle structure Δ_1 on a non-elementary manifold M_1 with $\rho_1(\Delta_1) = 1$, which is a contradiction. If no essential disc D exists as above, then we can find one which cuts the bridges in exactly three components. One readily enumerates the possible handle structures that satisfy our assumptions of valence-3 1-handles and non-peripheral disc lakes. After applying the hollowing out procedure, a Mom-2 handle structure is created. Of the two choices of which 1-handle is to be hollowed out, one will produce a full Mom-2 structure.

□

Remark 4.6. As an example of the type of non-full Mom-2 discussed in the previous lemma, note that the figure-8 knot complement contains a non-full Mom-2 structure with two 1-handles of valence 3, in addition to having a full Mom-2 structure.

Proof of Theorem 4.1 The proof is by induction on $\rho_1(\Delta)$. If (M, T, Δ) is not hyperbolic, then apply Lemma 3.11 to obtain (M_1, T_1, Δ_1) where $M_1 \to N$ is a non-elementary embedding, ∂M_1 is a union of tori and either (M_1, T_1, Δ_1) is a full internal Mom-k structure, $k < n$, or $\rho_1(\Delta_1) + 2 \leq \rho_1(\Delta)$. In the former case the proof follows by induction. In the latter case as seen in the next paragraph we will produce a general based internal Mom-k structure (M_4, B_4, Δ_4) with $k \leq n-2$ and M_4 hyperbolic. Lemma 4.4 then implies that $k \geq 2$ and hence $n \geq 4$. If $k = 2$, then Lemma 4.5 implies that N has a full internal Mom-2 structure (M_5, B_5, Δ_5). M_5 must be hyperbolic, or else the above arguement for $n = 2$ will give a contradiction to Lemma 4.4, completing the proof.

So assume that (M_1, T_1, Δ_1) is not a full internal Mom structure. If M_1 is reducible, then split along a normal reducing 2-sphere, retain the non-elementarily embedded component, cap off the resulting 2-sphere boundary component with a 3-handle and cancel that 3-handle with a 2-handle. After a sequence of such operations we obtain (M_2, T_2, Δ_2) with $\rho_1(\Delta_2) \leq \rho_1(\Delta_1)$ and M_2 irreducible. If M_2 contains an embedded essential torus R, then split along R and retain the non-elementarily embedded component. After a sequence of such operations we obtain (M_3, T_3, Δ_3) with $\rho_1(\Delta_3) \leq \rho_1(\Delta_2)$ and M_3 is irreducible and geometrically atoroidal. Since M_3 is non-elementarily embedded in N, it is not a Seifert fibered space and hence is anannular and so by Thurston [Th2] it is hyperbolic. After repeatedly applying Lemmas 4.2 and 4.3 we obtain a general based internal Mom-k structure (M_4, T_4, Δ_4) on N where M_4 is hyperbolic and $\rho_1(\Delta_4) \leq \rho_1(\Delta_3)$. □

5. Enumeration of hyperbolic Mom-n's for $2 \leq n \leq 4$

Let (M, T, Δ) be a full hyperbolic Mom-n, with $2 \leq n \leq 4$. The handle structure Δ collapses to a cellular complex K in the following fashion. Each 1-handle collapses to the arc at its core, and each 2-handle collapses to the disc at its core (expanded as necessary so that it is still attached to the cores of the appropriate 1-handles). Also, $T \times I$ collapses to $T \times 1$, subdivided into 0-cells, 1-cells, and 2-cells corresponding to the islands, bridges, and lakes of (M, T, Δ). (Note that if (M, T, Δ) were not full, we might have a non-simply connected lake and K would not be a proper cellular complex.)

The resulting complex K is a *spine* for M in the sense of [MF]. If all of the 1-handles of Δ are of valence 3, then it is also a *special spine* in the sense of [MF]; however K is not a special spine in general. In particular, in a special spine the link of each point is either a circle or a circle with two or three radii, but if Δ has a 1-handle of valence n then the endpoints of the corresponding arc in K will have links which are a circle with n radii. This, however, is the only way in which K fails to be a special spine.

In section 2 of [MF] Matveev and Fomenko describe how a manifold with a special spine can be reconstructed by gluing together truncated or ideal simplices dual to the vertices of the spine. This construction is easily generalized to our situation, and shows that M can be reconstructed from K by gluing together ideal polyhedra dual to the vertices of K. The result is an ideal cellulation of M which is dual to the cellular complex K.

The 3-cells of this cellulation will be dual to the elements of K^0, which consist of the endpoints of the cores of the 1-handles of Δ. In addition, since we've assumed each 1-handle of Δ meets at least two 2-handles, each point $v \in K^0$ will be the endpoint of at least two curves in $T \times 1 \cap K^1$. Hence if n_v is the valence of v in the 1-skeleton of K then $n_v \geq 3$. If $n_v \geq 4$ then v is dual to an $(n_v - 1)$-sided pyramid: the base of the pyramid is dual to the core of a 1-handle while the sides are dual to curves in $T \times 1 \cap K^1$. If $n_v = 3$ then v is dual to a "digonal pyramid", which we eliminate from the cellulation by collapsing it to a face in the obvious fashion. Thus K is dual to a cellulation of M by ideal pyramids. Since the bases of these pyramids correspond to the ends of the 1-handles of Δ, we can pair them up into a collection of ideal dipyramids.

We can say more concerning the possible types and combinations of dipyramids. On one hand, each vertex v is adjacent to $n_v - 1$ edges in $T \times 1 \cap K^1$, and each such edge has two endpoints; on the other hand, the core of each 2-cell of Δ contributes three edges to $T \times 1 \cap K^1$, and there are n such cores in a Mom-n. Therefore $\Sigma_v(n_v - 1) = 6n$ in a Mom-n. Furthermore, $n_v - 1$ must be at least 2 and (if it's greater than 2) equals the number of sides of the pyramid dual to v. Finally the vertices v occur in pairs since each one corresponds to an end of a 1-handle, and the vertices in each pair have the same valence. Therefore for a Mom-2, there are only two possibilities: four three-sided pyramids, which glue together to form two three-sided dipyramids, or two four-sided pyramids and two "digonal pyramids", which (after eliminating the "digonal pyramids") glue together to form a single ideal octahedron. Similarly, there are only three possibilities for a Mom-3: three three-sided dipyramids, a three-sided dipyramid together with an octahedron, or a five-sided dipyramid by itself. The five possibilities for a Mom-4 are: four three-sided dipyramids, two three-sided dipyramids and an octahedron, one three-sided dipyramid and one five-sided dipyramid, two octahedra, or one six-sided dipyramid.

Thus, if (M, T, Δ) is a hyperbolic Mom-2, Mom-3, or Mom-4 then M can be obtained by gluing together the faces of one of these ten sets of ideal polyhedra. Enumerating the possibilities for M then becomes a matter of enumerating the ways in which the faces of these polyhedra can be glued together to form a hyperbolic 3-manifold.

This task is simplified somewhat by the following observation: the faces of each dipyramid always have exactly one vertex which is dual to the cusp neighborhood $T \times [0, 1)$. When gluing the polyhedra together to form N, all such vertices must be identified with one another and with no other vertices. Thus given any two faces, there is only one possible orientation-preserving way that those two faces could be glued together.

Hence it is sufficient to enumerate the number of ways in which the faces of one of the ten sets of polyhedra can be identified in pairs. Although it is almost trivial to program a computer to do this, care must be taken as the number of possibilities is a factorial function of the number of faces, and a naive approach can rapidly exhaust a computer's memory. To reduce the demands on the computer, a refinement to the naive approach was employed. First, for each possible set of polyhedra a symmetry group was computed. Each dipyramid has dihedral symmetry, while if a given set of polyhedra contains two dipyramids with the same number of sides then they can be exchanged to provide an additional symmetry. Secondly, an ordering was chosen for the set of all possible pairings of faces, namely the lexicographic ordering of the pairings when represented as permutations. Our computer program considered the set of pairings in order, and any pairing was immediately rejected if it was conjugate to a previous pairing via an element of the symmetry group. This considerably reduced the running time of the program.

The next step in the process is to eliminate pairings which result in obviously non-hyperbolic manifolds. While the program SnapPea can in principle handle this, for reasons of speed our program checked one necessary criterion itself: whether the link of every ideal vertex was Euclidean. Computing the Euler characteristic of the link of each ideal vertex in the cellular complex resulting from a pairing was easy to do and eliminated many cases from consideration. Our program also eliminated any pairing in which the vertices supposedly dual to the original cusp neighborhood or solid torus in fact glued together to form two or more ideal vertices.

The above considerations resulted in a list of gluing descriptions of 4,231 manifolds which might be hyperbolic Mom-2's or Mom-3's. At this point, SnapPea was employed to try and compute hyperbolic metrics for each of these manifolds, and to find further hyperbolic symmetries among the manifolds which admitted such metrics. SnapPea claimed to find hyperbolic metrics in 164 cases. In three of those cases SnapPea had experienced an obvious floating-point error and "found" a hyperbolic metric with an absurdly low volume. In those three cases the programs Regina and GAP (which, unlike SnapPea, do not rely on floating-point arithmetic) were used to calculate the fundamental groups of the corresponding manifolds. In all three cases the fundamental group was isomorphic to the group $\langle a, b | [a, b^n] \rangle$ where $n = 3$ or 5, which has a non-trivial center. Therefore, these three manifolds could not possibly be hyperbolic and were rejected. That left 161 cases, which SnapPea identified as belonging to a total of 21 isometry classes of hyperbolic manifolds.

Some comments about rigor are in order here. Since SnapPea relies on floating-point arithmetic, some of its results are unavoidably inexact. In particular, there is no guarantee that SnapPea will find a hyperbolic metric on a manifold even if one exists, or that SnapPea will correctly discern the absence of a hyperbolic metric in cases where it doesn't exist. In practice it is our experience that if one is careful to allow SnapPea to simplify a triangulation before attempting to find a metric, then if a metric exists SnapPea will either find it or fail to make a determination, while if a metric doesn't exist SnapPea will either correctly say so or on rare occasions "find" a metric with absurdly low volume due to floating-point error. Still, from a standpoint of rigor this is problematic. Fortunately there is at least one task which SnapPea does perform exactly, and that is finding isometries between two different cusped manifolds: SnapPea will only report that an isometry exists if it finds identical triangulations of the two manifolds. (See [We]; in particular see the comments in the source code file *isometry.c*.) This is a combinatorial operation, not a floating-point one, and hence we are confident that SnapPea performs this operation rigorously.

Those familiar with SnapPea's source code may object that SnapPea re-triangulates each manifold before determining if an isometry exists, and that SnapPea uses floating-point information to choose the re-triangulation. To this objection we would reply that while floating-point information is used to *choose* the re-triangulation, the actual re-triangulating is still a combinatorial operation, i.e. it uses integer arithmetic. The new triangulation is guaranteed to have the same topological type (see the comments in *canonize_part_1.c* from [We]), and hence the possibility of floating-point error does not invalidate the result when SnapPea reports that it has found an isometry.

Thus while we are trusting SnapPea when it says that the 161 manifolds mentioned above are all in the isometry class of one of 21 manifolds from the SnapPea census, we are confident that we are not sacrificing rigor in so doing. Furthermore, the census manifolds were recently confirmed to be hyperbolic by Harriet Moser in [Mos], establishing that we have found 21 different hyperbolic Mom-2's and Mom-3's.

Unfortunately, we still can't trust SnapPea when it fails to find an hyperbolic metric for a given manifold, as that result is not guaranteed to be rigorous. This means that there are 4,067 manifolds from the above list of 4,231 which may still be hyperbolic despite SnapPea evidence to the contrary. These manifolds were analyzed separately in the same way as the three manifolds for which SnapPea claimed to have found an absurdly low hyperbolic volume. Namely, we used Regina and GAP as before to compute the fundamental groups of the manifolds in question, and then examined the list of groups to see if any of them might be the fundamental group of a hyperbolic manifold. The vast majority of the groups on the list either had a non-trivial center, or else had two rank-2 Abelian subgroups which intersected in a rank-1 Abelian subgroup (also impossible in the fundamental group of a hyperbolic 3-manifold). Some of the groups required further analysis but were still eventually rejected; for example, many groups had an index-two subgroup with one of the above properties even when it was not clear that the whole group had such properties.

In the end the hand analysis did not reveal any new hyperbolic 3-manifolds in the list of gluing descriptions. This completes the proof of the following:

Theorem 5.1. *There are 3 hyperbolic manifolds M such that (M, T, Δ) is a Mom-2 for some T and Δ: the manifolds known in SnapPea's notation as m125, m129, and m203. There are 18 additional hyperbolic manifolds M such that (M, T, Δ) is a Mom-3 structure for some T and Δ: the manifolds known in SnapPea's notation as m202, m292, m295, m328, m329, m359, m366, m367, m391, m412, s596, s647, s774, s776, s780, s785, s898, and s959.*

Some comments about this list are in order. The manifold m129, better known as the complement of the Whitehead link, is the only manifold on this list which is obtained by gluing together the faces of an ideal octahedron. Also, all but one of these manifolds have two cusps. The exception is the three-cusped s776, which

m357	s579	s883	v2124	v2943	v3292	v3450
m388	s602	s887	v2208	v2945	v3294	v3456
s441	s621	s895	v2531	v3039	v3376	v3468
s443	s622	s906	v2533	v3108	v3379	v3497
s503	s638	s910	v2644	v3127	v3380	v3501
s506	s661	s913	v2648	v3140	v3383	v3506
s549	s782	s914	v2652	v3211	v3384	v3507
s568	s831	s930	v2731	v3222	v3385	v3518
s569	s843	s937	v2732	v3223	v3393	v3527
s576	s859	s940	v2788	v3224	v3396	v3544
s577	s864	s941	v2892	v3225	v3426	v3546
s578	s880	s948	v2942	v3227	v3429	

FIGURE 1. Conjectured list of SnapPea manifolds which are strict Mom-4's.

is the complement in S^3 of a three-element chain of circles (the link 6^3_1 in Rolfsen's notation).

Enumerating hyperbolic Mom-4's was more difficult: merely enumerating the possible gluing descriptions resulted in a list of 1,033,610 possibilities (compared to 4,231 possibilities in the previous case). From this list, SnapPea identified 138 different hyperbolic manifolds. In another 493 cases, SnapPea was either unable to make a determination or else experienced an obvious floating-point error. In each such case, the fundamental group of the corresponding manifold was again computed by Regina and GAP, and in each case the fundamental group was isomorphic to $\langle a, b | [a^n, b] \rangle$ for some n, or else had two rank-2 Abelian subgroups which intersected in a rank-1 Abelian subgroup. Therefore these exceptional cases do not correspond to hyperbolic manifolds. Note that all of the Mom-2's and Mom-3's appear in the Mom-4 list; the same manifold can admit multiple handle structures.

Based on the above result, we propose the following:

Conjecture 5.2. *There are 138 hyperbolic manifolds M such that (M, T, Δ) is a Mom-2, Mom-3, or Mom-4 for some T and Δ. Of these, 117 are* strict *Mom-4's, i.e. Mom-4's which are not Mom-2's or Mom-3's.*

Of the 117 strict Mom-4's, SnapPea was successfully used to identify 83 of them as manifolds from the SnapPea census. Those manifolds appear in Figure 1. SnapPea was not able to identifiy the remaining 34 manifolds, and in fact 33 of those manifolds have volumes which do not appear anywhere in the SnapPea census, presumably because the Matveev complexity of the corresponding manifolds is greater than 7 (see [MF]). The remaining manifold has the same volume and homology as the census manifold v3527; it is conceivable that SnapPea was simply unable to find a corresponding isometry.

The unidentified manifolds are listed in Figure 2. The notation used can be interpreted as follows: the numbers before the semi-colon describe the type of ideal polyhedra used to construct the manifold. For example, the first entry in the figure has the numbers "3, 3, 4" to the left of the semi-colon; each "3" indicates an ideal triangular dipyramid, while each "4" indicates an ideal square dipyramid (i.e. an ideal octahedron). Each ideal dipyramid has two "polar" vertices and either three

(3, 3, 4 ; 3, 6, 8, 0, 13, 19, 1, 15, 2, 17, 14, 18, 16, 4, 10, 7, 12, 9, 11, 5)
(3, 3, 4 ; 3, 6, 11, 0, 10, 9, 1, 15, 14, 5, 4, 2, 16, 18, 8, 7, 12, 19, 13, 17)
(3, 3, 4 ; 3, 6, 11, 0, 9, 18, 1, 19, 13, 4, 15, 2, 16, 8, 17, 10, 12, 14, 5, 7)
(3, 3, 3, 3 ; 3, 6, 9, 0, 13, 19, 1, 22, 14, 2, 17, 12, 11, 4, 8, 23, 18, 10, 16, 5, 21, 20, 7, 15)
(3, 3, 3, 3 ; 3, 4, 6, 0, 1, 19, 2, 9, 13, 7, 14, 15, 23, 8, 10, 11, 20, 21, 22, 5, 16, 17, 18, 12)
(3, 3, 3, 3 ; 3, 6, 12, 0, 17, 9, 1, 11, 18, 5, 23, 7, 2, 20, 15, 14, 21, 4, 8, 22, 13, 16, 19, 10)
(3, 3, 4 ; 3, 4, 6, 0, 1, 18, 2, 11, 16, 10, 9, 7, 15, 19, 17, 12, 8, 14, 5, 13)
(3, 3, 4 ; 3, 4, 6, 0, 1, 18, 2, 14, 12, 13, 19, 17, 8, 9, 7, 16, 15, 11, 5, 10)
(3, 3, 4 ; 3, 4, 6, 0, 1, 18, 2, 14, 16, 13, 19, 17, 15, 9, 7, 12, 8, 11, 5, 10)
(3, 3, 4 ; 3, 6, 10, 0, 13, 9, 1, 16, 19, 5, 2, 17, 14, 4, 12, 18, 7, 11, 15, 8)
(3, 3, 4 ; 3, 6, 10, 0, 8, 13, 1, 16, 4, 18, 2, 17, 15, 5, 19, 12, 7, 11, 9, 14)
(3, 3, 4 ; 3, 6, 10, 0, 8, 13, 1, 18, 4, 16, 2, 19, 15, 5, 17, 12, 9, 14, 7, 11)
(4, 4 ; 15, 10, 13, 8, 11, 14, 9, 12, 3, 6, 1, 4, 7, 2, 5, 0)
(4, 4 ; 15, 14, 5, 6, 9, 2, 3, 10, 13, 4, 7, 12, 11, 8, 1, 0)
(4, 4 ; 15, 14, 9, 8, 11, 10, 13, 12, 3, 2, 5, 4, 7, 6, 1, 0)
(4, 4 ; 15, 4, 13, 6, 1, 8, 3, 10, 5, 14, 7, 12, 11, 2, 9, 0)
(4, 4 ; 15, 14, 4, 5, 2, 3, 11, 10, 12, 13, 7, 6, 8, 9, 1, 0)
(4, 4 ; 15, 14, 6, 7, 11, 10, 2, 3, 12, 13, 5, 4, 8, 9, 1, 0)
(4, 4 ; 15, 7, 13, 10, 9, 14, 11, 1, 12, 4, 3, 6, 8, 2, 5, 0)
(4, 4 ; 15, 5, 13, 7, 9, 1, 11, 3, 14, 4, 12, 6, 10, 2, 8, 0)
(3, 3, 4 ; 3, 6, 10, 0, 15, 17, 1, 18, 14, 16, 2, 19, 13, 12, 8, 4, 9, 5, 7, 11)
(3, 3, 4 ; 3, 6, 11, 0, 8, 19, 1, 15, 4, 17, 14, 2, 16, 18, 10, 7, 12, 9, 13, 5)
(3, 3, 4 ; 6, 7, 10, 8, 13, 17, 0, 1, 3, 15, 2, 19, 16, 4, 18, 9, 12, 5, 14, 11)
(3, 3, 3, 3 ; 3, 4, 6, 0, 1, 9, 2, 15, 17, 5, 13, 18, 19, 10, 23, 7, 22, 8, 11, 12, 21, 20, 16, 14)
(3, 3, 4 ; 3, 6, 10, 0, 8, 14, 1, 16, 4, 18, 2, 17, 13, 12, 5, 19, 7, 11, 9, 15)
(3, 3, 4 ; 3, 6, 10, 0, 8, 14, 1, 18, 4, 16, 2, 19, 13, 12, 5, 17, 9, 15, 7, 11)
(3, 3, 4 ; 3, 6, 7, 0, 16, 19, 1, 2, 10, 12, 8, 14, 9, 18, 11, 17, 4, 15, 13, 5)
(3, 3, 4 ; 6, 10, 19, 8, 13, 17, 0, 12, 3, 15, 1, 16, 7, 4, 18, 9, 11, 5, 14, 2)
(3, 3, 4 ; 6, 7, 10, 8, 9, 13, 0, 1, 3, 4, 2, 19, 16, 5, 17, 18, 12, 14, 15, 11)
(3, 3, 4 ; 3, 6, 10, 0, 8, 18, 1, 14, 4, 16, 2, 19, 13, 12, 7, 17, 9, 15, 5, 11)
(3, 3, 3, 3 ; 3, 6, 12, 0, 9, 16, 1, 18, 23, 4, 20, 22, 2, 19, 15, 14, 5, 21, 7, 13, 10, 17, 11, 8)
(3, 3, 3, 3 ; 3, 4, 6, 0, 1, 9, 2, 15, 17, 5, 14, 13, 19, 11, 10, 7, 23, 8, 22, 12, 21, 20, 18, 16)
(3, 3, 3, 3 ; 3, 6, 12, 0, 9, 16, 1, 10, 18, 4, 7, 22, 2, 20, 15, 14, 5, 21, 8, 23, 13, 17, 11, 19)
(3, 3, 3, 3 ; 3, 6, 12, 0, 9, 16, 1, 18, 11, 4, 23, 8, 2, 19, 15, 14, 5, 21, 7, 13, 22, 17, 20, 10)

FIGURE 2. Conjectured Mom-4's not identified by SnapPea.

or four "equatorial vertices". Number the faces of all the polyhedra sequentially in such a way that the faces "north" of each equator are numbered before the faces "south" of each equator. For example, in the first entry the first triangular dipyramid has faces 0, 1, and 2 next to one polar vertex, and faces 10, 11, and 12 next to the other polar vertex. The next triangular dipyramid has faces 3, 4, and 5 as well as faces 13, 14, 15, and the square dipyramid has faces 6 through 9 and 16 through 19. (This somewhat unintuitive numbering scheme was chosen for convenience when writing the computer software for this part of the paper.) Then the numbers to the right of the semi-colon form a permutation which describes how to glue together the faces of the ideal polyhedra. For example, in the first entry the string of numbers which begins with "3, 6, 8, 0, . . . " imply that face 0 is glued to face 3, face 1 is glued to face 6, and so on. Since we are requiring "polar" vertices to be identified solely with other "polar" vertices, no other information is needed to reconstruct the polyhedral gluing.

One additional point of information: all but eight of the manifolds in the list satisfy $|\partial M| = 2$; seven satisfy $|\partial M| = 3$ and one satisfies $|\partial M| = 4$. Thanks to the timely assistance of Morwen Thistlethwaite, the authors were able to positively identify all eight of these manifolds:

Conjecture 5.3. *There are 8 hyperbolic manifolds M such that M, T, Δ is a Mom-n for some $2 \leq n \leq 4$ and $|\partial M| > 2$. All eight manifolds are complements of links in S^3: the links 6^3_1, 6^3_2 (the Borromean rings), 8^3_1, 8^3_9, 8^4_2, and the links with Gauss codes jcccddEGHiJBFCa, jcbecceaHbIJDGF, and mccdfiEhAjKLcmdbFG.*

At the time of writing we are still searching for an efficient way to verify SnapPea's computations in the Mom-4 case; clearly, examining over a million fundamental groups by hand is not a practical solution. Until a better way is found, our enumeration results in the Mom-4 case should properly be considered speculative.

6. Hyperbolic Mom-n's in hyperbolic 3-manifolds are internal Mom-n structures for $n \leq 4$

Let R be a convolotube in the interior of a compact hyperbolic 3-manifold N and let V be the cube with knotted hole bounded by R. By drilling out solid tori from $N - \mathring{V}$, we can create a manifold M which is non-elementarily embedded in N and whose boundary contains a convolutube. We call such an embedding *knotted*. The goal of this chapter is to show that if $n \leq 4$, any embedding of a Mom-n manifold (M, T) into a compact hyperbolic manifold (N, T) is unknotted.

Definition 6.1. Let M be a compact 3-manifold and T a possibly empty union of components of ∂M. We say that (M, T) is *hereditarily unknotted*, if *every* non-elementary embedding into a compact hyperbolic 3-manifold N, taking T to components of ∂N, has the property that each component of ∂M is either boundary parallel or bounds a solid torus.

Remark 6.2. If (M, T) is herditarily unknotted and M_1 is obtained by filling a component of $\partial M - T$, then (M_1, T) is hereditarily unknotted.

Lemma 6.3. *If (M, T) is a hereditarily unknotted Mom-n manifold non-elementarily embedded in the hyperbolic 3-manifold N such that T bounds a tubular neighborhood of a geodesic, then (M, T) is an internal Mom-n structure.*

Proof. Let V be the solid torus bounded by T. By Lemma 1.7, if $N_1 = N - \mathring{V}$ with cusp neighborhoods deleted, then N_1 is compact hyperbolic. Therefore $(M, T) \subset (N_1, T)$ is a non-elementary embedding and hence any component of $\partial M - T$ either bounds a solid torus or is boundary parallel in N_1. Therefore similar properties hold in N and hence (M, T) is an internal Mom-n structure on N. □

Remark 6.4. The condition that T bounds a neighborhood of a geodesic is essential.

Lemma 6.5. *Let M be a compact hyperbolic 3-manifold with T a union of components of ∂M. If $\partial M - T$ is connected, (M, T) is hereditarily unknotted.*

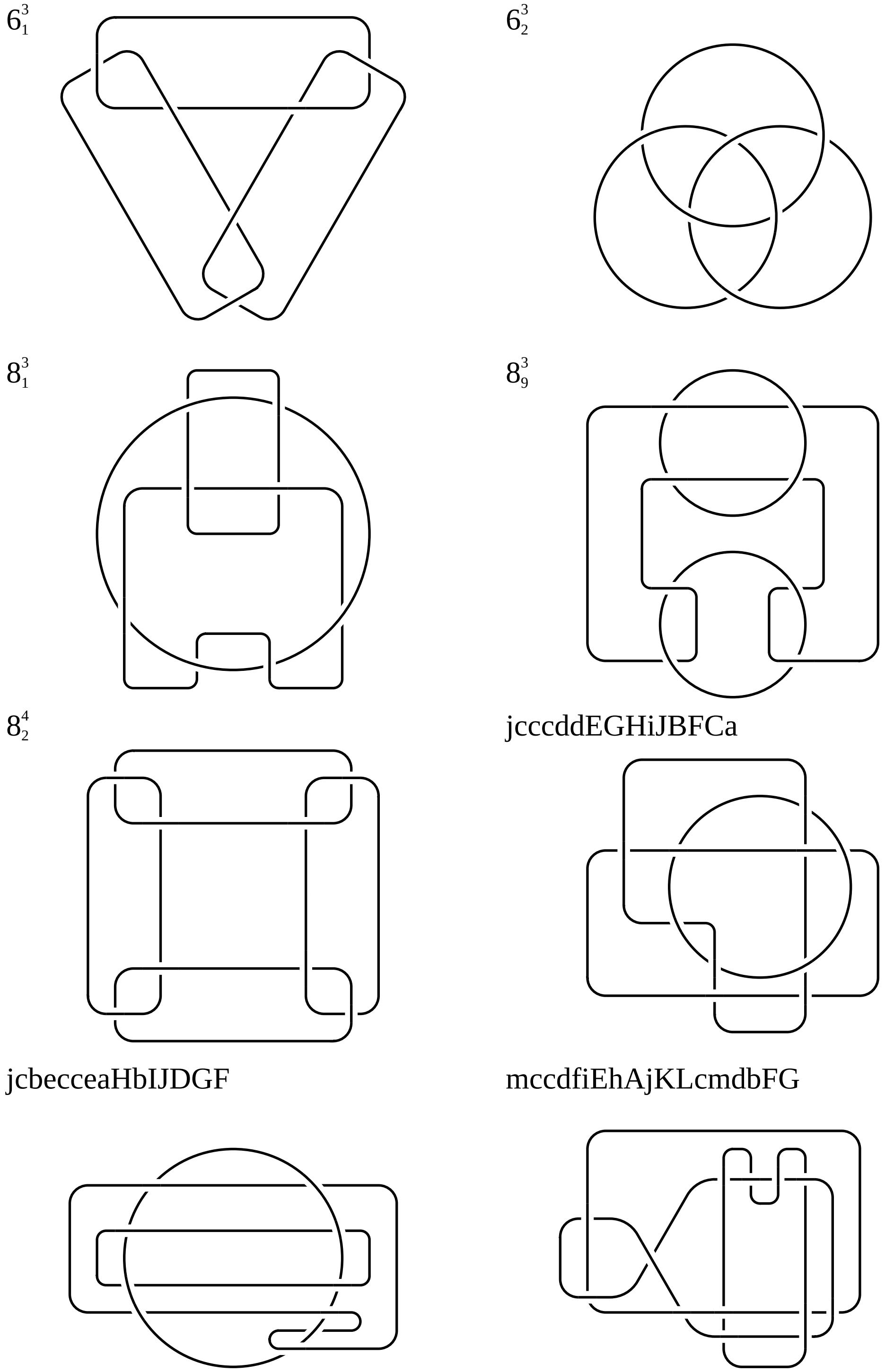

FIGURE 3. Eight links whose complements are Mom-4's with 3 or more cusps.

Proof. If under a non-elementary embedding $(M,T) \to (N,T)$, $\partial M - T$ was a convolutube, then M would be reducible. □

The following result establishes criteria for showing that (M,T) is hereditarily unknotted.

Lemma 6.6. *Let M be a compact hyperbolic 3-manifold with $V_1, \ldots, V_n$ components of ∂M and T a nonempty union of some other components. If any of the following hold, there exists no non-elementary embedding $(M,T) \to (N,T)$ such that N is compact hyperbolic and $\{V_1, \ldots, V_n\}$ is exactly the set of convolutubes of $\partial M \subset N$.*

i) The manifold obtained by some filling of M along $V_1, \ldots, V_n$ is a 3-manifold without any hyperbolic part. (That is, after applying sphere and torus decompositions there are no hyperbolic components.)

ii) After some filling of M along $V_1, \ldots, V_n$, the surface T is compressible.

iii) For every filling on a non-empty set of components of $\partial M - T \cup V_1 \cup \cdots \cup V_n$, either $V_1 \cup \cdots \cup V_n$ is incompressible or the filled manifold has no hyperbolic part.

Proof. Suppose that (M,T) embeds in (N,T), where among the components of ∂M, $V_1, \cdots, V_n$ are the set of convolutubes and $W_1, \cdots, W_m$ are the tubes. Let W_i^* denote the solid torus bounded by W_i and V_i^* denote the cube with knotted hole bounded by V_i. Let $B_1, \cdots, B_n$ be pairwise disjoint 3-balls in N such that for each i, $V_i \subset B_i$.

i) Let $\hat{M}$ be a manifold obtained by filling the V_i's. Let $\hat{N}$ be obtained by deleting the V_i^*'s and doing the corresponding fillings along the V_i's. Therefore $\hat{N}$ is obtained from $\hat{M}$ by Dehn filling and $\hat{N}$ is a connected sum of N with $S^2 \times S^1$'s and/or lens spaces and/or S^3's. This implies that $\hat{M}$ has a hyperbolic part.

ii) If T is compressible in $\hat{M}$ it is compressible in $\hat{N}$ and hence in N, which is a contradiction.

iii) First observe that V_i compresses in the manifold M' obtained by filling M where each W_i is filled with W_i^*. Topologically, M' is homeomorphic to N with n open unknotted and unlinked solid tori removed and so has a hyperbolic part. □

Theorem 6.7. *If the Mom-n manifolds for $n \leq 4$ with three or more boundary components are exactly those listed in Figure 3 (i.e. if Conjecture 5.1 is true), then any hyperbolic Mom-n manifold (M,T) with $n \leq 4$ is hereditarily unknotted.*

Proof. By Lemma 6.5 it suffices to consider the case where M is one of the eight Mom-4 manifolds with at least three boundary components listed in Figure 3. If M is any of the first six manifolds and T is any component of ∂M, then (M,T) is hereditarily unknotted by criterion (i) of Lemma 6.6. For manifolds 7 and 8, depending on which boundary component is used for T, applications of (i) and (iii) imply that they are hereditarily unknotted. □

7. Examples of Mom-n structures

In this section we give some representative examples of hyperbolic manifolds N which contain an internal Mom-2 or Mom-3 structure (M,T,Δ). Our goal in this section is to give the reader an intuitive feel for how these particular cell complexes arise inside hyperbolic manifolds. All of the manifolds in this section involve manifolds N with torus boundary, with the base torus of the Mom-structure (M,T,Δ) being ∂N. To obtain Mom-n structures on closed manifolds, note that if

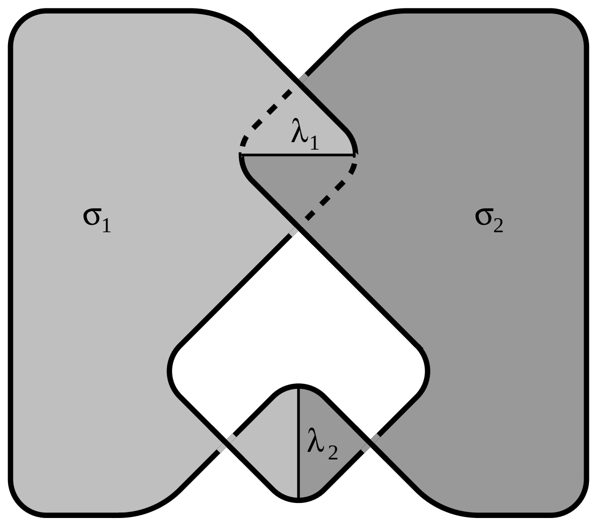

FIGURE 4. The figure-8 knot complement equipped with a Mom-2.

$T = \partial N$ then a Mom-n structure (M, T, Δ) on N passes to a Mom-n structure on any manifold obtained by filling ∂N.

Example 7.1. The first example is the figure-8 knot complement. We construct a Mom-2 (M, T, Δ) inside this manifold as follows. The torus T is just the boundary of the manifold. The 1-handles λ_1 and λ_2 span the two tangles which make up the standard diagram of this knot, as seen in figure 4. Finally the 2-handles σ_1 and σ_2 are symmetrically placed as shown in the diagram. Note that, as required, each 2-handle meets three 1 handles counting multiplicity. Specifically, each 2-handle meets λ_1 twice and λ_2 once. Also, one can see from the diagram that the complement of $T \cup \{\lambda_i\} \cup \{\sigma_j\}$ consists of a solid torus, and that the solid torus retracts onto a homotopically non-trivial simple closed curve (which is a geodesic in N). Thus this is a valid hyperbolic Mom-2 structure on N.

Moreover, we can quickly determine the nature of the ideal triangulation of M described in the Section 5. The ends of λ_1 are each dual to a four-sided pyramid in this triangulation, and the two endpoints of λ_2 are each dual to a "digonal pyramid", so that each get eliminated. Thus the figure-8 knot complement possesses a Mom-2 structure (M, T, Δ) where M is a two-cusped hyperbolic manifold which is in turn obtained by gluing together the faces of an ideal octahedron. By the comments after Theorem 5.1, M must be the complement of the Whitehead link. And indeed, it is easy to verify that if one drills out the core of the solid torus in the complement of M one obtains a manifold homeomorphic to the complement of the Whitehead link.

Example 7.2. Next we will let N be the manifold known as m003 in the SnapPea census. This manifold has first homology group $\mathbb{Z} + \mathbb{Z}/5$, and hence is not a knot complement; instead, we will present this manifold as the union of two regular ideal hyperbolic tetrahedra; see figure 5. Note that in the diagram each face is glued to the corresponding face on the other tetrahedron, in such a way that the edges match

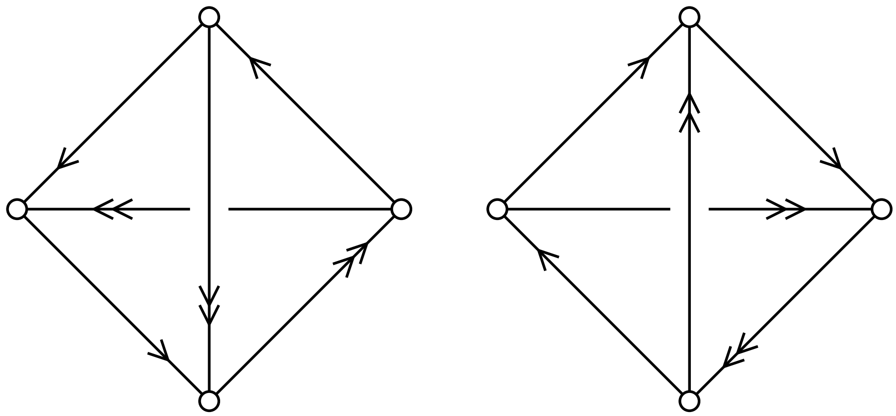

FIGURE 5. The two ideal tetrahedra making up the manifold m003.

up into two equivalence classes as shown. To make N a compact manifold with torus boundary, assume the ideal tetrahedra are truncated. Now suppose that we construct (M, T, Δ) in this case as follows. For the 1-handles, we use neighborhoods of the two edges shown in the diagram, truncated by the torus $T = \partial N$. And for the 2-handles, we use neighborhoods of the two truncated triangles which are formed by gluing together the faces on the front of each tetrahedron in the diagram. It is a simple exercise to confirm that the complement of the resulting embedded manifold M consists of a solid torus, and that the solid torus retracts onto a simple closed geodesic curve, and that therefore this manifold possesses a valid hyperbolic Mom-2 structure. Each of the 1-handles in this Mom-2 meets three of the 2-handles, counting multiplicity; therefore we can conclude that m003 contains a Mom-2 (M, T, Δ) where M is obtained by gluing together two ideal three-sided dipyramids. From Theorem 5.1 and the comments following it we know this must be either m125 or m203. Further investigation with SnapPea shows that it must in fact be m125.

It is instructive to get another view of this Mom-2 by constructing a cusp diagram for this manifold. Specifically, consider the triangulation induced on T by the given ideal triangulation of m003. The two ideal tetrahedra in m003 will appear as eight triangles, the four ideal triangles will appear as twelve edges, and the two edges will appear as four vertices. The resulting cusp diagram is shown in figure 6; keep in mind this is a diagram of a torus, so the edges of the parallelogram are identified with one another. (The labels inside each triangle indicate which of the ideal simplices contributes that triangle to the cusp diagram.) The highlighted edges in the cusp diagram are those that correspond to the 2-handles of the handle structure Δ; in other words, they along with the four vertices of the diagram comprise $\Delta^1 \cap T$.

Example 7.3. As another example in this vein, consider the manifold N=m017. This manifold has first homology group $\mathbb{Z}+\mathbb{Z}/7$, so again it is not a knot complement in S^3, but for brevity's sake we only present a cusp diagram here. In figure 7, the corners of the three ideal hyperbolic tetrahedra which make up m017 can be seen. And again, the highlighted edges in the cusp diagram correspond to two faces of those tetrahedra which provide the 2-handles for an internal Mom-2 in this manifold. Note that we can determine from the cusp diagram alone that

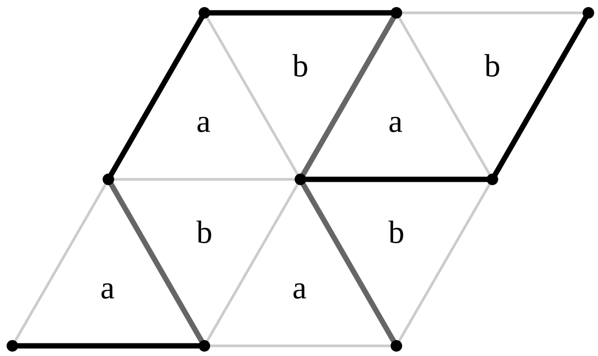

FIGURE 6. The cusp diagram for m003, with the components of the Mom-2 highlighted.

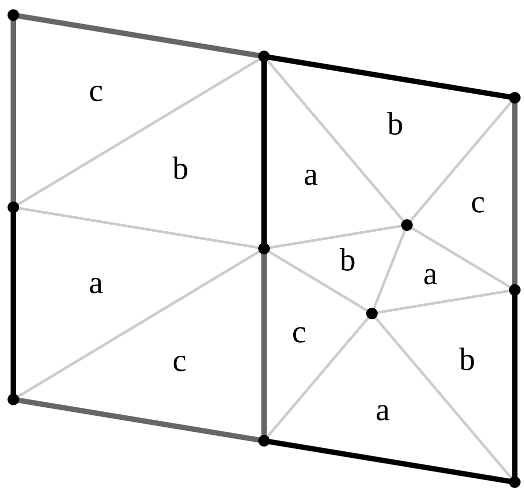

FIGURE 7. The cusp diagram for m017, with the components of a Mom-2 highlighted.

the 1-handles of this Mom-2 meet four and two 2-handles respectively, counting multiplicity, and that therefore in the resulting Mom-2 structure (M, T, Δ) the manifold M is obtained by gluing together an ideal octahedron. As before, this implies that M must be homeomorphic to the complement of the Whitehead link. Some further work with SnapPea confirms this: m017 is obtained by (-7,2) Dehn surgery on either component of the link.

Example 7.4. Finally, we include the motivating example for this paper. Figure 8 shows the maximal cusp diagram of the 1-cusped manifold m011 as provided by Weeks' SnapPea program. Unlike the previous cusp diagrams in this section, it also shows all the horoballs at hyperbolic distance at most 0.51 from the maximal horoball at infinity. The paralellogram shows a fundamental domain for the $\mathbb{Z} \oplus \mathbb{Z}$ action. Note that the ideal triangulation presented in this diagram is dual to the Ford decomposition of the manifold. In particular the 1-simplices of the triangulation are geodesics orthogonal to pairs of horoballs; these 1-simplices appear either

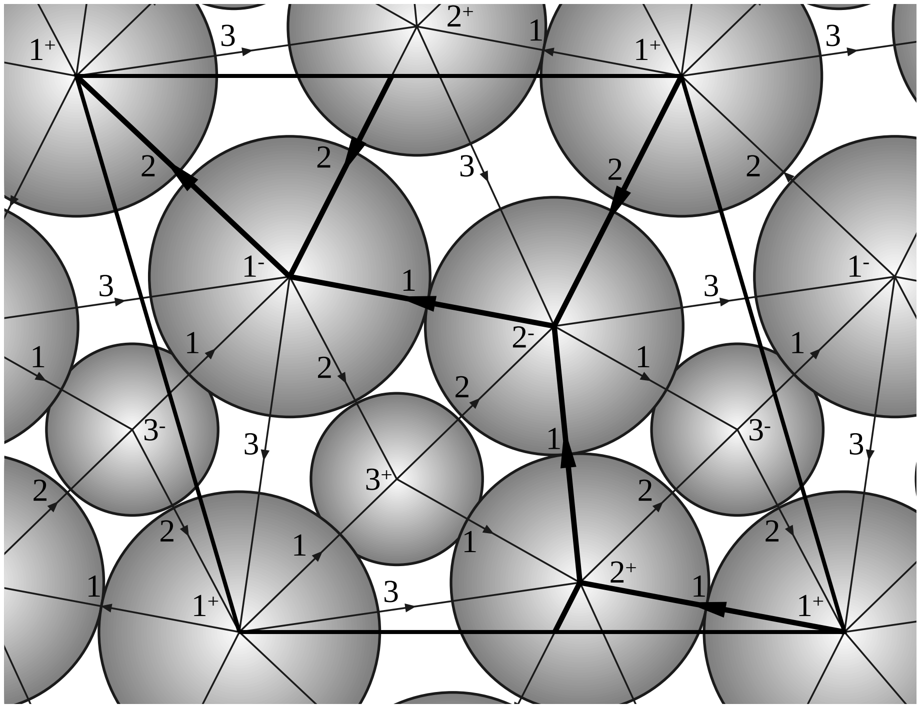

FIGURE 8. SnapPea's cusp diagram for m011, with the components of a Mom-2 highlighted.

as edges in the figure joining the endpoints of the simplex in S^2_∞, or as the vertical geodesics passing from the center of each horoball to the horoball at infinity.

Let H_∞ denote the horoball at infinity. There are six horoballs in the diagram up to the $\mathbb{Z}\oplus\mathbb{Z}$-action, labelled $1-$, $1+$, $2-$, $2+$, $3-$, and $3+$. This notation means that if $\gamma \in \pi_1(m011)$ takes horoball $n\pm$ to H_∞, the horoball at infinity, then $H\infty$ is transformed to one labelled $n\mp$. The geodesic from $n-$ to H_∞ is oriented to point into H_∞ and hence the geodesic from $n+$ to H_∞ is oriented out of H_∞. These orientations induce, via the $\pi_1(m011)$-action, the indicated orientations on the edges of the diagram. We explain, by example, the meaning of the edge labels. The edge 3 from $2+$ to $2-$ corresponds to a geodesic σ with the property that when $2+$ is transformed to H_∞, then $2-$ is transformed to $3+$ and σ is transformed to the vertical geodesic oriented from H_∞ to $3+$. (Had the edge been oriented oppositely, then $2-$ would have been transformed to $3-$.) SnapPea did not provide the orientation information, however such information can be derived from the SnapPea data.

By staring at this diagram we can see how m011 contains a Mom-2. Let V_0 be the maximal horotorus neighborhood of the cusp, slightly shrunken. By expanding V_0, the expanded V_0 touches the (expanded) horoballs labeled 1, thereby creating a 1-handle denoted E_1. Let V_1 denote this expanded V_0. Further expansion creates V_2 which is topologically V_1 together with another 1-handle E_2, this 1-handle occuring between horoball 2 and H_∞. The edge labelled 1 between horoballs $2-$ and $2+$ corresponds to a valence three 2-handle which goes over E_1 once and E_2 twice.

Similary the edge labelled 2 between horoballs $1-$ and $1+$ gives rise to a valence three 2-handle going twice over E_1 and once over E_2. The parallelogram of Figure 8 can also be viewed as ∂V_0, with the centers of $1-, 1+, 2-, 2+$ as the attaching sites of the 1-handles and the thick black lines corresponding to where the 2-handles cross over ∂V_0.

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