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ON THE TOPOLOGY OF ENDING LAMINATION SPACE

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Abstract. We show that if S is a finite type orientable surface of genus g and p punctures where $3g+p\geq 5$, then $\mathcal{EL}(S)$ is $(n-1)$-connected and $(n-1)$-locally connected, where $\dim(\mathcal{PML}(S))=2n+1=6g+2p-7$. Furthermore, if $g=0$, then $\mathcal{EL}(S)$ is homeomorphic to the $p-4$ dimensional Nobeling space.

0. Introduction

This paper is about the topology of the space $\mathcal{EL}(S)$ of *ending laminations* on a finite type hyperbolic surface, i.e. a complete hyperbolic surface S of genus-g with p punctures. An ending lamination is a geodesic lamination $\mathcal{L}$ in S that is minimal and filling, i.e. every leaf of $\mathcal{L}$ is dense in $\mathcal{L}$ and any simple closed geodesic in S nontrivally intersects $\mathcal{L}$ transversely.

Since Thurston's seminal work on surface automorphisms in the mid 1970's, laminations in surfaces have played central roles in low dimensional topology, hyperbolic geometry, geometric group theory and the theory of mapping class groups. From many points of view, the ending laminations are the most interesting laminations. For example, the stable and unstable laminations of a pseudo Anosov mapping class are ending laminations [Th1] and associated to a degenerate end of a complete hyperbolic 3-manifold with finitely generated fundamental group is an ending lamination [Th4], [Bon]. Also, every ending lamination arises in this manner [BCM].

The Hausdorff metric on closed sets induces a metric topology on $\mathcal{EL}(S)$. Here, two elements $\mathcal{L}_1$, $\mathcal{L}_2$ in $\mathcal{EL}(S)$ are *close* if each point in $\mathcal{L}_1$ is close to a point of $\mathcal{L}_2$ and vice versa. In 1988, Thurston [Th2] showed that with this topology $\mathcal{EL}(S)$ is totally disconnected and in 2004 Zhu and Bonahon [ZB] showed that with respect to the Hausdorff metric, $\mathcal{EL}(S)$ has Hausdorff dimension zero.

It is the *coarse Hausdorff topology* that makes $\mathcal{EL}(S)$ important for applications and gives $\mathcal{EL}(S)$ a very interesting topological structure. This is the topology on $\mathcal{EL}(S)$ induced from that of $\mathcal{PML}(S)$, the space of projective measured laminations of S. Let $\mathcal{FPML}(S)$ denote the subspace of $\mathcal{PML}(S)$ consisting of those measured laminations whose underlying lamination is an ending lamination. (Fact: every ending lamination fully supports a measure.) Then $\mathcal{EL}(S)$ is a quotient of $\mathcal{FPML}(S)$ and is topologized accordingly. Equivalently, a sequence $\mathcal{L}_1,\mathcal{L}_2,\cdots$ converges to $\mathcal{L}$ in the coarse Hausdorff topology if after passing to subsequence, the sequence converges in the Hausdorff topology to a diagonal extension of $\mathcal{L}$, i.e. a lamination obtained by adding finitely many leaves, [Ha]. From now on $\mathcal{EL}(S)$ will have the coarse Hausdorff topology.

In 1999, Erica Klarreich [K] showed that $\mathcal{EL}(S)$ is the Gromov boundary of $\mathcal{C}(S)$, the curve complex of S. (See also [H1].) As a consequence of many results

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in hyperbolic geometry e.g. [Ber], [Th3], [M], [BCM], [Ag], [CG]; Leininger and Schleimer [LS] showed that the space of doubly degenerate hyperbolic structures on $S \times \mathbb{R}$ is *homeomorphic* to $\mathcal{EL}(S) \times \mathcal{EL}(S) \setminus \Delta$, where Δ is the diagonal. For other applications see [RS] and §18.

If S is the thrice punctured sphere, then $\mathcal{EL}(S) = \emptyset$. If S is the 4-punctured sphere or once-punctured torus, then $\mathcal{EL}(S) = \mathcal{FPML}(S) = \mathbb{R} \setminus \mathbb{Q}$. In 2000 Peter Storm conjectured that if S is not one of these exceptional surfaces, then $\mathcal{EL}(S)$ is connected. Various partial results on the connectivity and local connectivity of $\mathcal{EL}(S)$ were obtained by Leininger, Mj and Schleimer, [LS], [LMS].

Using completely different methods, essentially by bare hands, we showed [G1] that if S is neither the 3 or 4-punctured sphere nor the once-punctured torus, then $\mathcal{EL}(S)$ is path connected, locally path connected, cyclic and has no cut points. We also asked whether ending lamination spaces of sufficiently complicated surfaces were n-connected.

Here are our two main results.

Theorem 0.1. *Let S be a finite type hyperbolic surface of genus-g and p-punctures. Then $\mathcal{EL}(S)$ is $(n-1)$-connected and $(n-1)$-locally connected, where $2n+1 = \dim(\mathcal{PML}(S)) = 6g+2p-7$.*

Theorem 0.2. *Let S be a $(4+n)$-punctured sphere. Then $\mathcal{EL}(S)$ is homeomorphic to the n-dimensional Nobeling space.*

Remark 0.3. Let T denote the compact surface of genus-g with p open discs removed and S the p-punctured genus-g surface. It is well known that there is a natural homeomorphism between $\mathcal{EL}(S)$ and $\mathcal{EL}(T)$. In particular, the topology of $\mathcal{EL}(S)$ is independent of the hyperbolic metric and hence all topological results about $\mathcal{EL}(S)$ are applicable to $\mathcal{EL}(T)$. Thus the main results of this paper are purely topological and applicable to compact orientable surfaces.

The m-dimensional Nobeling space $\mathcal{R}^{2m+1}_m$ is the space of points in $\mathcal{R}^{2m+1}$ with at most m rational coordinates. In 1930 G. Nobeling [N] showed that the m-dimensional Nobeling space is a universal space for m-dimensional separable metric spaces, i.e. any m-dimensional separable metric space embeds in $\mathcal{R}^{2m+1}_m$. This extended the work of his mentor K. Menger [Me] who in 1926 defined the m-dimensional Menger spaces M^{2m+1}_m, showed that the Menger curve is universal for 1-dimensional compact metric spaces and suggested that M^{2m+1}_m is universal for m-dimensional compact metric spaces. It is known to experts (e.g. see [Be]) that any map of a compact $\le m$-dimensional space into M^{2m+1}_m can be approximated by an embedding. This was generalized by Nagorko [Na] who proved that any map of a $\le m$-dimensional complete separable metric space into $\mathcal{R}^{2m+1}_m$ is approximatible by a closed embedding.

A recent result of Ageev [Av], Levin [Le] and Nagorko [Na] gave a positive proof of a major long standing conjecture characterizing the m-dimensional Nobeling space. (The analogous conjecture for Menger spaces was proven by Bestvina [Be] in 1984.) Nagorko [HP] recast this result to show that the m-dimensional Nobeling space is one that satisfies a series of topological properties that are discussed in detail in §7, e.g. the space is $(m-1)$-connected, $(m-1)$-locally connected and satisfies the *locally finite m-discs property.* To prove Theorem 16.1 we will show that $\mathcal{EL}(S)$ satisfies these conditions for $m = n$.

Using [G1], Sebastian Hensel and Piotr Przytycki [HP] earlier showed that if S is either the 5-punctured sphere or twice-punctured torus, then $\mathcal{EL}(S)$ is homeomorphic to the one dimensional Nobeling space. They also boldly conjectured that all ending lamination spaces are homeomorphic to Nobeling spaces.

Independently, in 2005 Bestvina and Bromberg [BB] asked whether ending lamination spaces are Nobeling spaces. They were motivated by the fact that Menger spaces frequently arise as boundaries of locally compact Gromov hyperbolic spaces, Klarreich's theorem and the fact that the curve complex is not locally finite.

Theorem 0.2 gives a positive proof of the Hensel - Przytycki conjecture for punctured spheres. In section §19 we offer various conjectures on the topology of ending lamination spaces including three duality conjectures relating the curve complex with ending lamination space.

The methods of this paper are essentially also by bare hands. There are two main difficulties that must be overcome to generalize the methods of [G1] to prove Theorem 0.1. First of all it is problematic getting started. To prove path connectivity, given $\mu_0, \mu_1 \in \mathcal{EL}(S)$ we first chose $\lambda_0, \lambda_1 \in \mathcal{PML}(S)$ such that $\phi(\lambda_i) = \mu_i$ where ϕ is the forgetful map. The connectivity of $\mathcal{PML}(S)$ implies there exists a path $\mu : [0,1] \to \mathcal{PML}(S)$ such that $\mu(0) = \mu_0$ and $\mu(1) = \mu_1$. In [G1] we found an appropriate sequence of generic such paths, projected them into lamination space, and took an appropriate limit which was a path in $\mathcal{EL}(S)$ between μ_0 and μ_1. To prove simple connectivity, say for $\mathcal{EL}(S)$ where S is the surface of genus 2, the first step is already problematic, for there is a simple closed curve γ in $\mathcal{EL}(S)$ whose preimage in $\mathcal{PML}(S)$ does not contain a loop projecting to γ. (See Theorem 18.7.) In the general case, the preimage is a Cech-like loop and that turns out to be good enough. The second issue is that points along a generic path in $\mathcal{PML}(S)$ project to laminations that are *almost filling almost minimal*, a property that was essential in order to take limits in [G1]. In the general case, the analogous laminations are not close to being either almost filling or almost minimal. To deal with this we introduce the idea of *markers* that is a technical device that enables us to take limits of laminations with the desired controlled properties.

This paper basically accomplishes two things. It shows that if $k \leq n$, then any generic PL map $f : B^k \to \mathcal{PML}(S)$ can be *ϵ-approximated* by a map $g : B^k \to \mathcal{EL}(S)$ and conversely any map $g : B^k \to \mathcal{EL}(S)$ can be *ϵ-approximated* by a map $f : B^k \to \mathcal{PML}(S)$. Here $\dim(\mathcal{PML}(S)) = 2n + 1$. See section §13 for the precise statements.

In §1 we provide some basic information and facts about ending lamination space, point out an omission in §7 [G1] and prove the following. See [LMS] and [LS] for earlier partial results.

Theorem 0.4. *If S is a finite type hyperbolic surface that is not the 3 or 4-holed sphere or 1-holed torus, then $\mathcal{FPML}(S)$ is connected.*

In §2 we show that if $g : S^k \to \mathcal{EL}(S)$ is continuous, then there exists a continuous map $F : B^{k+1} \to \mathcal{PMLEL}(S)$ that extends g. Here $\mathcal{PMLEL}(S)$ is the disjoint union of $\mathcal{PML}(S)$ and $\mathcal{EL}(S)$ appropriately topologized. In §3 we introduce *markers*. In §4 we give more facts relating the topologies of $\mathcal{EL}(S)$ and $\mathcal{PML}(S)$. In §5 we give a criterion for a sequence $f_1, f_2, \cdots$ of maps of B^{k+1} into $\mathcal{PMLEL}(S)$ that restrict to $g : S^k \to \mathcal{EL}(S)$ to *converge* to a continuous map $G : B^{k+1} \to \mathcal{EL}(S)$, extending g. The core technical work of this paper is carried out in §6-10. In §11-12 we prove Theorem 0.1. In §13 we isolate out our $\mathcal{PML}(S)$ and $\mathcal{EL}(S)$ approximation theorems. In §14 we develop a theory of good cellulation sequences of $\mathcal{PML}(S)$, which may be of independent interest. In §15 we give various upper and lower estimates of $\dim(\mathcal{EL}(S))$ and prove that $\dim(S_{0,n+4}) = n$ and that $\pi_n(S_{0,n+4}) \neq 0$. In §16 we state Nagorko's recharacterization of Nobeling spaces. In §17 we prove that $S_{0,n+4}$ satisfies the locally finite n-discs property. In §18 various applications are given. In §19 we offer some problems and conjectures.

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1. Basic definitions and facts

In what follows, S or $S_{g,p}$ will denote a complete hyperbolic surface of genus g and p punctures. We will assume that the reader is familiar with the basics of Thurston's theory of curves and laminations on surfaces, e.g. $\mathcal{L}(S)$ the space of geodesic laminations with topology induced from the Hausdorff topology on closed sets, $\mathcal{ML}(S)$ the space of measured geodesic laminations endowed with the weak* topology, $\mathcal{PML}(S)$ projective measured lamination space, as well as the standard definitions and properties of of train tracks. For example, see [PH], [Ha], [M], [Th1], or [FLP]. All laminations in this paper will be compactly supported. See [G1] for various ways to view and measure distance between laminations as well as for standard notation. For example, $d_{PT(S)}(x, y)$ denotes the distance between the geodesic laminations x and y as measured in the projective unit tangent bundle. Unless said otherwise, distance between elements of $\mathcal{L}$ are computed via the Hausdorff topology on closed sets in $PT(S)$. Sections §1 and §2 (through Remark 2.5) of [G1] are also needed. Among other things, important aspects of the PL structure of $\mathcal{PML}(S)$ and $\mathcal{ML}(S)$ are described there.

Notation 1.1. We denote by $p : \mathcal{ML}(S) \setminus 0 \to \mathcal{PML}(S)$, the canonical projection and $\phi : \mathcal{PML}(S) \to \mathcal{L}(S)$ and $\hat{\phi} : \mathcal{ML}(S) \to \mathcal{L}(S)$, the forgetful maps. If τ is a train track, then $V(\tau)$ will denote the cell of measures supported on τ and $P(\tau)$ the polyhedron $p(V(\tau) \setminus 0)$.

Definition 1.2. Let $\mathcal{EL}(S)$ denote the set of ending laminations on S, i.e. the set of geodesic laminations that are *filling* and *minimal*. A lamination $\mathcal{L} \in \mathcal{L}(S)$ is *minimal* if every leaf is dense and *filling* if the metric closure (with respect to the induced path metric) of $S \setminus \mathcal{L}$ supports no simple closed geodesic.

The following is well known.

Lemma 1.3. *If $x \in \mathcal{PML}(S)$, then $\phi(x)$ is the disjoint union of minimal laminations. In particular $\phi(x) \in \mathcal{EL}(S)$ if and only if $\phi(x)$ is filling.*

Proof. Since the measure on x has full support, no non compact leaf L is proper, i.e. non compact leaves limit on themselves. It follows that $\phi(x)$ decomposes into a disjoint union of minimal laminations. If $\phi(x)$ is filling, then there is only one such component. $\square$

Notation 1.4. Let $\mathcal{FPML}(S)$ denote the subspace of $\mathcal{PML}(S)$ consisting of filling laminations and $\mathcal{UPML}(S)$ denote the subspace of unfilling laminations. Thus $\mathcal{PML}(S)$ is the disjoint union of $\mathcal{FPML}(S)$ and $\mathcal{UPML}(S)$.

Definition 1.5. Topologize $\mathcal{EL}(S)$ by giving it the quotient topology induced from the surjective map $\phi : \mathcal{FPML}(S) \to \mathcal{EL}(S)$ where $\mathcal{FPML}(S)$ has the subspace topology induced from $\mathcal{PML}(S)$. After [H1] we call this the *coarse Hausdorff topology*. Hamenstadt observed that this topology is a slight coarsening of the Hausdorff topology on $\mathcal{EL}(S)$; a sequence $\mathcal{L}_1, \mathcal{L}_2 \cdots$ in $\mathcal{EL}(S)$ limits to $\mathcal{L} \in \mathcal{EL}(S)$ if and only if any convergent subsequence in the Hausdorff topology converges to a diagonal extension $\mathcal{L}'$ of $\mathcal{L}$. A *diagonal extension* of $\mathcal{L}$ is a lamination obtained by adding finitely many leaves.

Remark 1.6. It is well known that $\mathcal{EL}(S)$ is separable and complete. Separability follows from the fact that $\mathcal{PML}(S)$ is a sphere and $\mathcal{FPML}(S)$ is dense in $\mathcal{PML}(S)$. (E.g. $\mathcal{FPML}(S)$ is the complement of countably many codimension-1 PL-cells in $\mathcal{PML}(S)$.) Masur and Minsky [MM] showed that the curve complex $\mathcal{C}(S)$ is Gromov-hyperbolic and Klarreich [K] (see also [H1]) showed that the Gromov boundary of $\mathcal{C}(S)$ is homeomorphic to $\mathcal{EL}(S)$ with the coarse Hausdorff topology. Being the boundary of a Gromov hyperbolic space, $\mathcal{EL}(S)$ is metrizable. Bonk and Schramm showed that with appropriate constants in the Gromov product, the induced Gromov metric is complete [BS]. See also [HP].

The following are characterizations of continuous maps in $\mathcal{EL}(S)$ analogous to Lemmas 1.13-1.15 of [G1]. Here X is a metric space.

Lemma 1.7. *A function $f : X \to \mathcal{EL}(S)$ is continuous if and only if for each $t \in X$ and each sequence $\{t_i\}$ converging to $t, f(t)$ is the coarse Hausdorff limit of the sequence $f(t_1), f(t_2), \cdots$.* □

Lemma 1.8. *A function $f : X \to \mathcal{EL}(S)$ is continuous if and only if for each $\epsilon > 0$ and $t \in X$ there exists a $\delta > 0$ such that $d_X(s,t) < \delta$ implies that the maximal angle of intersection between leaves of $f(t)$ and leaves of $f(s)$ is $< \epsilon$.* □

Lemma 1.9. *A function $f : X \to \mathcal{EL}(S)$ is continuous if and only if for each $\epsilon > 0$ and $t \in X$ there exists a $\delta > 0$ such that $d_X(s,t) < \delta$ implies that $d_{PT(S)}(f'(t),(f'(s)) < \epsilon$, where $f'(s)$ (resp $f'(t)$) is any diagonal extension of $f(s)$ (resp. $f(t)$).* □

The forgetful map $\phi : \mathcal{PML}(S) \to \mathcal{L}(S)$ is discontinuous, for any simple closed curve viewed as a point in $\mathcal{PML}(S)$ is the limit of filling laminations and any Hausdorff limit of a sequence of filling laminations is filling.

Definition 1.10. Let $X_1, X_2, \cdots$ be a sequence of subsets of the topological space Y. We say that the subsets $\{X_i\}$ *super converges* to X if for each $x \in X$, there exists $x_i \in X_i$ so that $\lim_{i\to\infty} x_i = x$. In this case we say X is a *sublimit* of $\{X_i\}$.

We will repeatedly use the following result that first appears in [Th4]. See Proposition 3.2 [G1] for a proof.

Proposition 1.11. *If the projective measured laminations $\lambda_1, \lambda_2, \cdots$ converge to $\lambda \in \mathcal{PML}(S)$, then $\phi(\lambda_1), \phi(\lambda_2), \cdots$ super converges to $\phi(\lambda)$ as subsets of $PT(S)$.*

The following consequence of super convergence, Proposition 1.11 is repeatedly used in this paper and in [G1].

Lemma 1.12. *If $z_1, z_2, \cdots$ is a convergent sequence in $\mathcal{EL}(S)$ limiting to z_∞ and $x_1, x_2, \cdots$ is a sequence in $\mathcal{PML}(S)$ such that for all i, $\phi(x_i) = z_i$, then any convergent subsequence of the x_i's converges to a point of $\phi^{-1}(z_\infty)$.* □

Proof. After passing to subsequence it suffices to consider the case that $x_1, x_2, \cdots$ coverges to x_∞ in $\mathcal{PML}(S)$ and that $z_1, z_2, \cdots$ converges to $\mathcal{L} \in \mathcal{L}(S)$ with respect to the Hausdorff topology. By super convergence, Proposition 1.11 $\phi(x_\infty)$ is a sublamination of $\mathcal{L}$ and by definition of coarse Hausdorff topology $\mathcal{L}$ is a diagonal extension of z_∞. Since z_∞ is minimal and each leaf of $\mathcal{L} \setminus z_\infty$ is non compact and proper it follows that $\phi(x_\infty) = z_\infty$. □

Corollary 1.13. *If $L \subset \mathcal{EL}(S)$ is compact, then $\phi^{-1}(L) \subset \mathcal{PML}(S)$ is compact.* □

Corollary 1.14. *$\phi : \mathcal{FPML}(S) \to \mathcal{EL}(S)$ is a closed map.* □

Lemma 1.15. *If $\mu \in \mathcal{EL}(S)$, $x_1, x_2, \cdots \to x$ is a convergent sequence in $\mathcal{PML}(S)$ and $\lim_{i\to\infty} d_{PT(S)}(\phi(x_i), \mu') = 0$ for some diagonal extension μ' of μ, then $x \in \phi^{-1}(\mu)$.*

Proof. After passing to subsequence and super convergence $\phi(x_1), \phi(x_2), \cdots$ converges to $\mathcal{L} \in \mathcal{L}(S)$ with respect to the Hausdorff topology where $\phi(x) \subset \mathcal{L}$. If $\phi(x) \neq \mu$, then $d_{PT(S)}(\mathcal{L}, \mu') > 0$ and hence $d_{PT(S)}(\phi(x_i), \mu')$ is uniformly bounded away from 0 for every diagonal extension μ' of μ. □

Lemma 1.16. *Let τ be a train track and $\mu \in \mathcal{EL}(S)$. Either τ carries μ or $\inf\{d_{PT(S)}(\phi(t), \mu) | t \in P(\tau)\} > 0$.*

Proof. If $\inf\{d_{PT(S)}(\phi(t), \mu) | t \in P(\tau)\} = 0$, then by compactness of $P(\tau)$ and the previous lemma, there exists $x \in P(\tau)$ such that $\phi(x) = \mu$ and hence τ carries μ. □

The following is well known, e.g. it can be deduced from Proposition 1.9 [G1].

Lemma 1.17. *If $z \in \mathcal{EL}(S)$, then $\phi^{-1}(z) = \sigma_z$ is a compact convex cell, i.e. if τ is any train track that carries z, then $\sigma_z = p(V)$, where $V \subset V(\tau)$ is the open cone on a compact convex cell in $V(\tau)$.* □

Theorem 1.18. *If S is a finite type hyperbolic surface that is not the 3 or 4-holed sphere or 1-holed torus, then $\mathcal{FPML}(S)$ is connected.*

Proof. If $\mathcal{FPML}(S)$ is disconnected, then it is the disjoint union of non empty closed sets A and B. By the previous lemma, if $\mathcal{L} \in \mathcal{EL}(S)$, then $\phi^{-1}(\mathcal{L})$ is connected. It follows that $\phi(A) \cap \phi(B) = \emptyset$ and hence by Lemma 1.14, $\mathcal{EL}(S)$ is the disjoint union of the non empty closed sets $\phi(A)$, $\phi(B)$ and hence $\mathcal{EL}(S)$ is disconnected. This contradicts [G1]. □

Remark 1.19. In the exceptional cases, $\mathcal{FPML}(S_{0,3}) = \emptyset$, $\mathcal{FPML}(S_{0,4}) = \mathcal{FPML}(S_{1,1}) = \mathbb{R} \setminus \mathbb{Q}$.

Definition 1.20. The curve complex $\mathcal{C}(S)$ introduced by Harvey [Ha] is the simplicial complex with vertices the set of simple closed geodesics and $(v_0, \cdots, v_p)$ span a simplex if the v_i's are pairwise disjoint.

There is a natural injective continuous map $\hat{I} : \mathcal{C}(S) \to \mathcal{ML}(S)$. If v is a vertex, then $I(v)$ is the measured lamination supported on v with transverse measure

$1/\operatorname{length}(v)$. Extend $\hat{I}$ linearly on simplices. Define $I : \mathcal{C}(S) \to \mathcal{PML}(S)$ by $I = p \circ \hat{I}$.

Remark 1.21. The map I is not a homeomorphism onto its image. If $\mathcal{C}(S)_{\text{sub}}$ denotes the topology on $\mathcal{C}(S)$ obtained by pulling back the subspace topology on $I(\mathcal{C}(S))$, then $\mathcal{C}(S)_{\text{sub}}$ is coarser than $\mathcal{C}(S)$. Indeed, if C is a vertex, then there exists a sequence $C_0, C_1, \cdots$ of vertices that converges in $\mathcal{C}(S)_{\text{sub}}$ to C, but does not have any limit points in $\mathcal{C}(S)$. See §19 for some conjectures relating the topology of $\mathcal{C}(S)$ and $\mathcal{C}(S)_{\text{sub}}$.

Correction 1.22. In §7 of [G1] the author states without proof Corollary 7.4 which asserts that if S is not one of the three exceptional surfaces, then $\mathcal{EL}(S)$ has no cut points. It is not a corollary of the statement of Theorem 7.1 because as the figure-8 shows, cyclic does not imply no cut points. It is not difficult to prove Corollary 7.4 by extending the proof of Theorem 7.1 to show that given $x \neq y \in \mathcal{EL}(S)$, there exists a simple closed curve in $\mathcal{EL}(S)$ passing through both x and y. Alternatively, no cut points immediately follows from the locally finite 1-discs property, Proposition 17.8 and Remark 16.3.

Notation 1.23. If X is a space, then $|X|$ denotes the number of components of X. If X and Y are sets, then $X \setminus Y$ is X with the points of Y deleted, but if $X, Y \in \mathcal{L}(S)$, then $X \setminus Y$ denotes the union of leaves of X that are not in Y. Thus by Lemma 1.3 if $X = \phi(x)$, $Y = \phi(y)$ with $x, y \in \mathcal{PML}(S)$, then $X \setminus Y$ is the union of those minimal sublaminations of X which are not sublaminations of Y.

2. Extending maps of spheres into $\mathcal{EL}(S)$ to maps of balls into $\mathcal{PMLEL}(S)$

Definition 2.1. Let $\mathcal{PMLEL}(S)$ (resp. $\mathcal{MLEL}(S)$) denote the disjoint union $\mathcal{PML}(S) \cup \mathcal{EL}(S)$ (resp. $\mathcal{ML}(S) \cup \mathcal{EL}(S)$). Define a topology on $\mathcal{PMLEL}(S)$ (resp. $\mathcal{MLEL}(S)$) as follows. A basis consists of all sets of the form $U \cup V$ where $\phi^{-1}(V) \subset U$ (resp. $\hat{\phi}^{-1}(V) \subset U$) where U is open in $\mathcal{PML}(S)$ (resp. $\mathcal{ML}(S)$) and V (possibly $\emptyset$) is open in $\mathcal{EL}(S)$. We will call this the *PMLEL topology* (resp. *MLEL topology*).

Lemma 2.2. *The PMLEL (resp. MLEL) topology has the following properties.*

i) $\mathcal{PMLEL}(S)$ (resp. $\mathcal{ML}(S) \cup \mathcal{EL}(S)$) is non Hausdorff. In fact x and y cannot be separated if and only if $x \in \mathcal{EL}(S)$ and $y \in \phi^{-1}(x)$ (resp. $y \in \hat{\phi}^{-1}(x)$), or vice versa.

ii) $\mathcal{PML}(S)$ (resp. $\mathcal{ML}(S)$) is an open subspace.

iii) $\mathcal{EL}(S)$ is a closed subspace.

iv) If $U \subset \mathcal{PML}(S)$ is a neighborhood of $\phi^{-1}(x)$ where $x \in \mathcal{EL}(S)$, then there exists an open set $V \subset \mathcal{EL}(S)$, such that $x \in V$ and $\phi^{-1}(V) \subset U$, i.e. $U \cup V$ is open in $\mathcal{PMLEL}(S)$.

v) A sequence $x_1, x_2, \cdots$ in $\mathcal{PML}(S)$ converges to $x \in \mathcal{EL}(S)$ if and only if every limit point of the sequence lies in $\phi^{-1}(x)$. A sequence $x_1, x_2, \cdots$ in $\mathcal{ML}(S)$ bounded away from both 0 and ∞ converges to $x \in \mathcal{EL}(S)$ if and only if every limit point of the sequence lies in $\hat{\phi}^{-1}(x)$.

Proof. Parts i)-iii) and v) are immediate. Part iv) follows from Lemma 1.12. □

Definition 2.3. Let V be the underlying space of a finite simplicial complex. A *generic PL map* $f : V \to \mathcal{PML}(S)$ is a PL map transverse to each $B_{a_1} \cap \cdots B_{a_r} \cap \delta B_{b_1} \cap \cdots \cap \delta B_{b_s}$, where $a_1, \cdots, a_r, b_1, \cdots, b_s$ are simple closed geodesics. (Notation as in [G1].) More generally $f : V \to \mathcal{PMLEL}(S)$ is called a *generic PL map* if $f^{-1}(\mathcal{EL}(S)) = W$ is a subcomplex of V and $f|(V \setminus W)$ is a generic PL map.

Lemma 2.4. *Let $L = I(L_0)$ where L_0 is a finite q-subcomplex of $\mathcal{C}(S)$. If $f : V \to \mathcal{PMLEL}(S)$ is a generic PL-map where $\dim(V) = p$ and $p + q \leq 2n$, then $f(V) \cap L = \emptyset$. If either $p + q \leq 2n - 1$ or $p \leq n$, then for every simple closed geodesic C, $f(V) \cap Z = \emptyset$, where $Z = \phi^{-1}(C) * (\partial B_C \cap L)$.*

Proof. By genericity and the dimension hypothesis, the first conclusion is immediate. For the second, $L \cap \partial B_C$ is at most $\min(q, n-1)$ dimensional, hence any simplex in the cone $(L \cap \partial B_C) * C$ is at most $\min(q+1, n)$ dimensional. Thus the result again follows by genericity. □

Remark 2.5. In this paper, V will be either a B^k or $S^k \times I$ and $X = \partial B^k$ or $S^k \times 1$.

Notation 2.6. Fix once and for all a map $\psi : \mathcal{EL}(S) \to \mathcal{PML}(S)$ such that $\phi \circ \psi = id_{\mathcal{EL}(S)}$. If $i : \mathcal{PML}(S) \to \mathcal{ML}(S)$ is the map sending a projective measured geodesic lamination to the corresponding measured geodesic lamination of length 1, then define $\hat{\psi} : \mathcal{EL}(S) \to \mathcal{ML}(S)$ by $\hat{\psi} = i \circ \psi$. Let $p : \mathcal{ML}(S) \setminus 0 \to \mathcal{PML}(S)$ denote the standard projection map and let $\hat{\phi} = \phi \circ p$.

While ψ is very discontinuous, it is continuous enough to carry out the following which is the main result of this section.

Proposition 2.7. *Let $g : S^{k-1} \to \mathcal{EL}(S)$ be continuous where $k \leq \dim(\mathcal{PML}(S)) = 2n+1$. Then there exists a generic PL map $F : B^k \to \mathcal{PMLEL}(S)$ (resp. $\hat{F} : B^k \to \mathcal{MLEL}(S)$) such that $F|S^{k-1} = g$ (resp. $\hat{F}|S^{k-1} = g$) and $F(\text{int}(B^k)) \subset \mathcal{PML}(S)$ (resp. $\hat{F}(\text{int}(B^k)) \subset \mathcal{ML}(S)$).*

Idea of Proof for $\mathcal{PML}(S)$: It suffices to first find a continuous extension and then perturb to a generic PL map. To obtain a continuous extension F, consider a sequence $K_1, K_2, \cdots$ of finer and finer triangulations of S^{k-1}. For each i, consider the map $f_i : S^{k-1} \to \mathcal{PML}(S)$ defined as follows. If κ is a simplex of K_i with vertices $v_0, \cdots, v_m$, then define $f_i(v_j) = \psi(g(v_j))$ and extend f_i linearly on κ. Extend f_1 to a map of B^k into $\mathcal{PML}(S)$ and extend f_i and f_{i+1} to a map of $F_i : S^{k-1} \times [i, i+1]$ into $\mathcal{PML}(S)$. Concatenating these maps and taking a limit yields the desired continuous map $F : B^k \cup S^{k-1} \times [1, \infty] \to \mathcal{PMLEL}(S)$ where ∂B^k is identified with $S^{k-1} \times 1, G|S^{k-1} \times [i, i+1] = F_i$ and $H|S^n \times \infty = g$.

Remark 2.8. The key technical issue is making precise the phrase "extend f_i linearly on σ".

Before we prove the Proposition we establish some notation and then prove a series of lemmas.

Notation 2.9. Let Δ be the cellulation on $\mathcal{PML}(S)$ whose cells are the various $P(\tau_i)$'s, where the τ_i's are the standard train tracks to some fixed parametrized pants decomposition of S. If $\sigma = P(\tau_i)$ is a cell of Δ and $Y \subset \sigma$, then define the *convex hull* of Y to be $p(C(p^{-1}(Y)))$, where $C(Z)$ is the convex hull of Z in $V(\tau_i)$.

If $x \in \mathcal{EL}(S)$, then let τ_x denote the unique standard train track that fully carries x and σ_x the cell $P(\tau_x)$.

If $\mathcal{L} \in \mathcal{L}(S)$, then $\mathcal{L}'$ will denote a *diagonal extension of* $\mathcal{L}$, i.e. a lamination obtained by adding finitely many non compact leaves.

Remark 2.10. It follows from Lemma 1.17 that $\phi^{-1}(x) \subset \operatorname{int}(\sigma_x)$ and is closed and convex.

Lemma 2.11. *Let $x_1, x_2, \cdots \to x$ be a convergent sequence in $\mathcal{EL}(S)$. If $y \in \mathcal{PML}(S)$ is a limit point of $\{\psi(x_i)\}$, then $\phi(y) = x$.*

Proof. This follows from Lemma 1.12. ☐

Definition 2.12. Fix $\epsilon_1 < 1/1000(\min(\{d(\sigma, \sigma')|\sigma, \sigma'$ disjoint cells of $\Delta\}$. For each cell σ of Δ define a retraction $r_\sigma : N(\sigma, \epsilon_1) \to \sigma$. For every $\delta < \epsilon_1$ define a discontinuous map $\pi_\delta : \mathcal{PML}(S) \to \mathcal{PML}(S)$ as follows. Informally, π_δ retracts a closed neighborhood of δ^0 to δ^0, then after deleting this neighborhood retracts a closed neighborhood of Δ^1 to Δ^1, then after deleting this neighborhood retracts a closed neighborhood of Δ^2 to Δ^2 and so on. As $\delta \to 0$, the neighborhoods are required to get smaller. More formally, let $\delta(0) = \delta$. If σ is a 0-cell of Δ, then define $\pi_\delta|N(\sigma, \delta(0)) = r_v|(N(\sigma, \delta(0))$. Now choose $\delta(1) \leq \delta$ such that if σ, σ' are distinct 1-cells, then $N(\sigma, \delta(1)) \cap N(\sigma', \delta(1)) \subset N(\Delta^0, \delta)$. If σ is a 1-cell, then define $\pi_\delta|N(\sigma, \delta(1)) \setminus N(\Delta^0, \delta(0)) = r_\sigma|N(\sigma, \delta(1)) \setminus N(\Delta^0, \delta(0))$. Having defined π_δ on $N(\Delta^0, \delta(0)) \cup \cdots \cup N(\Delta^k, \delta(k))$, extend π_δ in a similar way over $N(\Delta^{k+1}, \delta(k+1))$ using a sufficiently small $\delta(k+1)$. We require that if $\delta < \delta'$, then for all $i, \delta(i) < \delta'(i)$.

Remark 2.13. Let $x \in \sigma$ a cell of Δ. If $\delta_1 < \delta_2 \leq \epsilon_1$ and σ_1, σ_2 are the lowest dimensional cells of Δ respectively containing $\pi_{\delta_1}(x), \pi_{\delta_2}(x)$, then σ_2 is a face of σ_1 which is a face of σ.

Lemma 2.14. *Let $f_i : B^k \to \mathcal{EL}(S)$ be a sequence of maps such that $f_i(B^k) \to x \in \mathcal{EL}(S)$ in the Hausdorff topology on closed sets in $\mathcal{EL}(S)$.*

i) Then for i sufficiently large $\psi(f_i(B^k)) \subset st(\sigma_x)$, the open star of σ_x in Δ.

ii) Given $\epsilon > 0$ and $\delta < \epsilon_1$, then for i sufficiently large, if $t \in C_i$, where C_i denotes the convex hull of $r_{\sigma_x}(\psi(f_i(B^k)))$ in σ_x, then $x \in d_{PT(S)}(\phi(t), \epsilon)$.

Proof. If i) is false, then after passing to subsequence, for each i there exists $a_i \in B^k$ such that $\psi(f_i(a_i)) \notin \operatorname{st}(\sigma_x)$. This contradicts Lemma 2.11 which implies that any limit point of $\psi(f_i(a_i))$ lies in $\phi^{-1}(x) \subset \sigma_x$.

This argument actually shows that if U is any neighborhood of $\phi^{-1}(x)$, then $\psi(f_i(B^k)) \subset U$ for i sufficiently large and hence $r_{\sigma_x}(\psi(f_i(B^k))) \subset U \cap \sigma_x$ for $i$sufficiently large. Since $\phi^{-1}(x)$ is convex it follows that $C_i \subset U$ for i sufficiently large. By super convergence there exists a neighborhood V of $\phi^{-1}(x)$ such that if $y \in V$, then $x \in d_{PT(S)}(\phi(y), \epsilon)$. Now choose U so that the $r_{\sigma_x}(U) \subset V$. ☐

Lemma 2.15. *If $\epsilon > 0$ and $x_1, x_2, \cdots \to x \in \mathcal{EL}(S)$, then there exists $\delta > 0$ such that for i sufficiently large $x \subset N_{PT(S)}(\phi(\pi_\delta(\psi(x_i))), \epsilon)$.*

Proof. Let U be a neighborhood of $\phi^{-1}(x)$ such that $\lambda \in U$ implies $x \subset N_{PT(S)}(\phi(\lambda), \epsilon)$. Next choose $\delta > 0$ so that $\pi_\delta(N(\phi^{-1}(x), \delta)) \subset U$. Now apply Lemma 2.11 to show that for i sufficiently large the conclusion holds. ☐

Lemma 2.16. *Let $\epsilon > 0, g_i : B^k \to \mathcal{EL}(S)$ and $g_i(B^k) \to x$. There exists $N \in \mathbb{N}, \delta > 0$, such that if $\delta_1, \delta_2 \leq \delta, i \geq N$ and σ a simplex of Δ, then if C_i is the convex hull of $\sigma \cap (\pi_{\delta_1}(\psi(g_i(B^k))) \cup \pi_{\delta_2}(\psi(g_i(B^k))) \cup (\psi(g_i(B^k))))$ and $t \in C_i$, then $x \subset N_{PT(S)}(\phi(t), \epsilon)$.*

Proof. Let V be a neighborhood of $\phi^{-1}(x)$ such that if $y \in V$, then $x \in N_{PT(S)}(\phi(y), \epsilon)$. Let $V_1 \subset V$ be a neighborhood of $\phi^{-1}(x)$ such that for each cell $\sigma \subset \Delta$ containing σ_x as a face, the convex hull of $V_1 \cap \sigma \subset V$. Also assume that $\bar{V}_1 \cap \kappa = \emptyset$ if κ is a cell of Δ disjoint from σ_x. Choose $\delta < \epsilon_1$ so that $d(\kappa, V_1) > 2\delta$ for all cells κ disjoint from σ_x. Choose N so that $i \geq N$ and $\delta' \leq \delta$, then $\pi_{\delta'}(g_i(B^k)) \subset V_1$ and $g_i(B^k) \subset V_1$. Therefore, if $\delta_1, \delta_2 < \delta$ and $i \geq N$ then C_i lies in V. □

We now address how to approximate continuous maps into $\mathcal{EL}(S)$ by PL maps into $\mathcal{PML}(S)$.

Definition 2.17. Let σ be a cell of Δ. Let κ be a p-simplex and $H^0 : \kappa^0 \to \sigma$, where κ^0 are the vertices of κ. Define the *induced map* $H : \kappa \to \sigma$, such that $H|\kappa^0 = H^0$ as follows. Let $\hat{H}$ be the linear map of κ into $\hat{\sigma} = p^{-1}(\sigma)$ such that $\hat{H}|\kappa^0 = i \circ H^0$. Then define $H = p \circ \hat{H}$. In a similar manner, if K is a simplicial complex and $h|K^0 \to \mathcal{PML}(S)$ is such that for each simplex $\kappa, h(\kappa^0) \subset \sigma$, for some cell σ of Δ, then h extends to a map $H : K \to \mathcal{PML}(S)$ also called the *induced map*. Since the linear structure on a face of a cell of $\mathcal{ML}(S)$ is the restriction of the linear structure of the cell, H is well defined.

Lemma 2.18. *Let $g : K_1 \to \mathcal{EL}(S)$ be continuous where K_1 is a finite simplicial complex. Let $K_1, K_2, \cdots$ be such that $mesh(K_i) \to 0$ and each K_{i+1} is a subdivision of K_i. For every $\delta < \epsilon_1$ there exists an $i(\delta) \in \mathbb{N}$, monotonically increasing as $\delta \to 0$, such that if $\delta' \leq \delta, i = i(\delta)$ and κ is a simplex of K_i, then $\pi_{\delta'}(\psi(g(\kappa))) \cup \pi_\delta(\psi(g(\kappa)))$ is contained in a cell of Δ.*

Given $\epsilon > 0$ there exist $\delta(\epsilon) > 0$ and $N(\epsilon) \in \mathbb{N}$ such that if $i = N(\epsilon)$, κ is a simplex of K_i, σ a simplex of $\Delta, \delta_1, \delta_2 \leq \delta(\epsilon)$ and C is the convex hull of $\sigma \cap (\pi_{\delta_1}(\psi(g(\kappa))) \cup \pi_{\delta_2}(\psi(g(\kappa))) \cup \psi(g(\kappa)))$, then given $z_1 \in \kappa, z_2 \in C_i$ we have $d_{PT(S)}(g(z_1), \phi(z_2)) < \epsilon$.

Fix $\epsilon > 0$. If δ is sufficiently small, $i > i(\delta)$ and $H_i : K_i \to \mathcal{PML}(S)$ is the induced map arising from $\pi_\delta \circ \psi \circ g|K_i^0$, then for each $z \in K_1, d_{PT(S)}(\phi(H_i(z)), g(z)) < \epsilon$.

Proof. Fix $0 < \delta < \epsilon_1$. For each $x \in \mathcal{EL}(S)$, there exists a neighborhood V_x of $\phi^{-1}(x)$ such that $\pi_\delta(V_x) \subset \sigma_x$. By Lemma 2.2 iv) there exists a neighborhood U_x of x such that $\psi(U_x) \subset V_x$. By compactness there exist $U_{x_1}, \cdots, U_{x_n}$ that cover $g(K)$. There exists $i(\delta) > 0$ such that if $i \geq i(\delta)$ and κ is a simplex of K_i, then $g(\kappa) \subset U_{x_j}$ for some j and so $\pi_\delta(\psi(g(\kappa))) \subset \sigma_{x_j}$. Since each $\phi^{-1}(x)$ lies in the interior of a cell, it follows that σ_{x_j} is the lowest dimensional cell of Δ that contains any point of $\pi_\delta(\psi(g(\kappa))$. Now let $\delta' < \delta$. As above, $\pi_{\delta'}(\psi(g(\kappa))) \subset \sigma_y$ for some $y \in g(K_1)$ where σ_y is a minimal dimensional cell. It follows from Remark 2.13 that σ_{x_j} is a face of σ_y.

The proof of the second conclusion follows from that of the first and the proof of Lemma 2.16.

If the third conclusion of the lemma is false, then after passing to subsequence there exist $(\kappa_1, \omega_{1_j}, \delta_1, t_1), (\kappa_2, \omega_{2_j}, \delta_2, t_2), \cdots$ such that for all i, κ_{i+1} is a codimension-0 subsimplex of κ_i which is a simplex of $K_i, \omega_{i_j} \subset \kappa_i$ is a simplex of K_{i_j} some

$i_j \geq i, \delta_i \to 0$ and for some $t_i \in \omega_{i_j}, d_{PT(S)}(\phi(H_{i_j}(t_i)), g(t_i)) > \epsilon$, where H_{i_j} is the induced map corresponding to δ_i and K_{i_j}. Also $1_j < 2_j < \cdots$. If $t = \cap_{i=1}^{\infty}\kappa_i$ and B is homeomorphic to κ_1, then there exists maps $g_i : B \to \mathcal{EL}(S)$ such that $g_i(B) = g(\kappa_i)$ and $\lim_{i\to\infty} g_i(B) = g(t)$. Let σ_i be a cell of Δ that contains $H_{i_j}(t_i)$. Since $H_{i_j}(t_i)$ lies in the convex hull of $\pi_{\delta_i}(\psi(g_i(B))) \cap \sigma_i$ it follows by Lemma 2.16 that for i sufficiently large $g(t) \subset N_{PT(S)}(\phi(H_{i_j}(t_i), \epsilon/2)$. Convergence in the coarse Hausdorff topology implies that for i sufficiently large if $z \in \kappa_i$ then $g(t) \subset N_{PT(S)}(g(z), \epsilon/2)$. Taking $z = t_i$ we conclude that $d_{PT(S)}(\phi(H_{i_j}(t_i), g(t_i)) < \epsilon$, a contradiction. □

Proof of Proposition 2.7: Let $K_1, K_2, \cdots$ be subdivisions of S^{k-1} such that $\mathrm{mesh}(K_i) \to 0$ and each K_{i+1} is a subdivision of K_i. Pick $\delta_j < \delta(1/j)$ and $n_j > \max\{i(\delta_j), N(1/j)\}$ such that for $i \geq n_j$ and $\delta \leq \delta_j$ the induced map $H_k : K_i \to \mathcal{PML}(S)$ arising from $\pi_\delta \circ \psi \circ g|K_i^0$ satisfies the third conclusion of Lemma 2.18. Assume that $n_1 < n_2 < \cdots$. Replace the original $\{K_i\}$ sequence by the subsequence $\{K_{n_i}\}$. With this new sequence, let $f_j : K_j \to \mathcal{PML}(S)$ be the induced map arising from $\pi_{\delta_j} \circ \psi \circ g|K_i^0$.

Apply Lemma 2.18 to find, after passing to a subsequence of the K_i's, a sequence of maps $f_j : K_j \to \mathcal{PML}(S)$ so that each f_j satisfies the conclusions of those results for $\epsilon = 1/j$. Let δ_j denote the δ used to define f_j. Note that with this new sequence of K_i's, each K_j satisfies the first two conclusions of Lemma 2.18 using δ_j.

Define a triangulation $\mathcal{T}$ on $S^{k-1} \times [0, \infty)$ by first letting $\mathcal{T}|S^{k-1} \times j = K_j$ and then extending in a standard way to each $S^{k-1} \times [j, j+1]$ so that if ζ is a simplex of $\mathcal{T}|S^{k-1} \times [j, j+1]$, then $\zeta^0 \subset (\kappa^0 \times j) \cup (\kappa_1^0 \times (j+1))$ where κ is a simplex of K_j and $\kappa_1 \subset \kappa$ is a simplex of K_{j+1}. By Lemma 2.18, $\pi_{\delta_j}(\psi(g(\kappa^0))) \cup \pi_{\delta_{j+1}}(\psi(g(\kappa_1^0)))$ lie in the same cell of Δ so the induced maps on $\mathcal{T}|S^{k-1} \times \{j, j+1\}$ extend to one called $f_{j,j+1}$ on $\mathcal{T}|S^{k-1} \times [j, j+1]$.

Since $k \leq \dim \mathcal{PML}(S)$, f_1 extends to a map f_1' of B^k into $\mathcal{PML}(S)$. Define $F : B^k \cup S^{k-1} \times [1, \infty]$ to $\mathcal{PMLEL}(S)$ so that $F|B^k = f_1', F|S^{k-1} \times [i, i+1] = f_{i,i+1}$ and $F|S^{k-1} \times \infty = g$. It remains to show that F is continuous at each $(z, \infty) \in S^{k-1} \times \infty$. Let $(z_1, t_1), (z_2, t_2), \cdots \to (z, \infty)$. By passing to subsequence we can assume that $\phi(F(z_i, t_i)) \to \mathcal{L} \in \mathcal{L}(S)$ where convergence is in the Hausdorff topology. If $\mathcal{L}$ is not a diagonal extension of $g(z)$, then $\mathcal{L}$ is transverse to $g(z)$ and hence $d_{PT(S)}(F(z_i, t_i), g(z)) > \epsilon$ for i sufficiently large and some $\epsilon > 0$. By passing to subsequence we can assume that $z_i \in \kappa_i$, where κ_i is a simplex in K_i, where $\kappa_1 \supset \kappa_2 \supset \cdots$ is a nested sequence of simplicies and $\cap \kappa_i = z$. Let n_i denote the greatest integer in t_i. Apply the second conclusion of Lemma 2.18 to $\kappa_i, \delta_{n_i}, \delta_{n_i+1}$ to conclude that $\lim_{i\to\infty} d_{PT(S)}(z, \phi(F(z_i, t_i))) = 0$, a contradiction. □

Remark 2.19. If there exists a train track τ such that each $z \in g(S^{k-1})$ is carried by τ and $V \subset P(\tau)$ is the convex hull of $\phi^{-1}(g(S^{k-1}))$, then there exists a continuous extension $F : B^k \to \mathcal{PMLEL}(S)$ such that $F(\mathrm{int}(B^k)) \subset V$. Indeed, since V is convex we can dispense with the use of the π_δ's and directly construct the maps $f_i, f_{i,i+1}, f_1'$ to have values within V.

The following local version is needed to prove local $(k-1)$-connectivity of $\mathcal{EL}(S)$, when $k \leq n$ and $\dim(\mathcal{PML}(S)) = 2n + 1$.

Proposition 2.20. *If $z \in \mathcal{EL}(S)$, then for every neighborhood U of $\phi^{-1}(z)$ there exists a neighborhood V of $\phi^{-1}(z)$ such that if $g : S^{k-1} \to \mathcal{EL}(S)$ is continuous and*

$\psi^{-1}(g(S^{k-1})) \subset V$, then there exists a generic PL map $F : B^k \to \mathcal{PMLEL}(S)$ such that $F|(\mathrm{int}(B^k)) \subset U$ and $F|S^{k-1} = g$.

Proof. There exists a parametrized pair of pants decomposition of S such that z is fully carried by a maximal standard train track τ. Thus $\hat{\phi}^{-1}(z)$ is a closed convex set in $\mathrm{int}(V(\tau)) - 0$. If $\hat{U} = p^{-1}(U)$, then $\hat{U} \cap (\mathrm{int}(V(\tau)))$ is a neighborhood of $\hat{\phi}^{-1}(z)$, since τ is maximal. Let $\hat{V} \subset \mathrm{int}(V(\tau))$ be a convex neighbohood of $\hat{\phi}^{-1}(z)$ saturated by open rays through the origin such that $\hat{V} \subset \hat{U}$. Let $V = p(\hat{V})$. Then V is a convex neighborhood of $\phi^{-1}(x)$ with $V \subset U$. By Remark 2.19 if $\psi(g(S^{k-1})) \subset V$, then there exists a continuous map $F : B^k \to \mathcal{PMLEL}(S)$ such that $F|S^{k-1} = g$ and $F(\mathrm{int}(B^k)) \subset V$. Now replace F by a generic perturbation. ☐

3. Markers

In this section we introduce the idea of a *marker* which is a technical device for controlling geodesic laminations in a hyperbolic surface. In the next section, using markers, we will show that under appropriate circumstances a sequence of maps $f_i : B^k \to \mathcal{PMLEL}(S)$, $i = 1, 2, \cdots$ extending a given continuous map $g : S^{k-1} \to \mathcal{EL}(S)$ converges to an extension $f_\infty : B^k \to \mathcal{EL}(S)$. As always, S will denote a finite type surface with a fixed complete hyperbolic metric.

Definition 3.1. Let α_0, α_1 be open embedded geodesic arcs in S. A *path* from α_0 to α_1 is a continuous map $f : [0, 1] \to S$ such that for $i = 0, 1, f(i) \subset \alpha_i$. Two paths are *path homotopic* if they are homotopic through paths from α_0 to α_1. Given two path homotopic paths f, g from α_0 to α_1, a lift $\tilde{\alpha}_0$ of α_0 to $\mathbb{H}^2$ determines unique lifts $\tilde{f}, \tilde{g}, \tilde{\alpha}_1$ respectively of f, g, α_1 so that $\tilde{f}, \tilde{g}$ are homotpic paths from $\tilde{\alpha}_0$ to $\tilde{\alpha}_1$. Define $d_{\tilde{H}}(f, g) = d_H(\tilde{f}(I), \tilde{g}(I))$, where d_H denotes Hausdorff distance measured in $PT(\tilde{S})$. Note that this is well defined independent of the lift of α_0.

Definition 3.2. A *marker* $\mathcal{M}$ for the hyperbolic surface S consists of two embedded (though not necessarily pairwise disjoint) open geodesic arcs α_0, α_1 called *posts* and a path homotopy class $[\alpha]$ from α_0 to α_1. A representative β of $[\alpha]$ is said to *span* $\mathcal{M}$. The marker $\mathcal{M}$ is an *ϵ-marker* if whenever β and β' are geodesics in S spanning $\mathcal{M}$, then $d_{\tilde{H}}(\beta, \beta') < \epsilon$ and $\mathrm{length}(\beta) \geq 1$.

Let C be a simple closed geodesic in S. A *C-marker* is a marker $\mathcal{M}$ such that if β is a geodesic arc spanning $\mathcal{M}$, then β is transverse to C and $|\beta \cap C| > 4g + p + 1$ where $g = \mathrm{genus}(S)$ and p is the number of punctures.

In a similar manner we define the notion of *closed ϵ* or *closed C-marker*. Here the posts are closed geodesic arcs. In this case the requirement $d_{\tilde{H}}(\beta, \beta') < \epsilon$ is replaced by $d_{\tilde{H}}(\beta, \beta') \leq \epsilon$. If $\mathcal{M}$ is an ϵ or C-marker, then $\bar{\mathcal{M}}$ will denote the corresponding closed ϵ or C-marker.

Definition 3.3. We say that the geodesic L hits the marker $\mathcal{M}$ if there exists ≥ 3 distinct embedded arcs in L that span $\mathcal{M}$. We allow for the possibility that distinct arcs have non trivial overlap. We say that the geodesic lamination $\mathcal{L}$ *hits* the marker $\mathcal{M}$ if there exists a leaf L of $\mathcal{L}$ that hits $\mathcal{M}$. If $b_1, \cdots, b_m$ are simple closed geodesics, then we say that $\mathcal{M}$ is $\mathcal{L}$-free of $\{b_1, \cdots, b_m\}$ if some leaf $L \notin \{b_1, \cdots, b_m\}$ of $\mathcal{L}$ hits $\mathcal{M}$.

Lemma 3.4. *Let S be a finite type hyperbolic surface with a fixed hyperbolic metric. Given $\epsilon > 0$ there exists $N(\epsilon) \in \mathbb{N}$ such that if β is an embedded geodesic arc,* *length*$(\beta) \le 2$ *and* $\mathcal{L} \in \mathcal{L}(S)$ *is such that* $|\mathcal{L} \cap \beta| > N(\epsilon)$, *then there exists an* ϵ*-marker* $\mathcal{M}$ *hit by* $\mathcal{L}$ *with posts* $\alpha_0, \alpha_1 \subset \beta$.

Proof. $\mathcal{L}$ has at most $6|\chi(S)|$ boundary leaves. Thus some leaf of $\mathcal{L}$ hits β at least $|\mathcal{L}\cap\beta|/(6|\chi(S)|)$ times. Since length(β) is uniformly bounded, if $|\mathcal{L}\cap\beta|$ is sufficiently large, then three distinct segments of some leaf must have endpoints in β, be nearly parallel and have length ≥ 2. Now restrict to appropriate small arcs of β to create α_0 and α_1 and let $[\alpha]$ be the class represented by the three segments. $\square$

Lemma 3.5. *If* $\mathcal{L} \in \mathcal{L}(S)$ *has a non compact leaf* L, *then for every* $\epsilon > 0$ *there exists an* ϵ*-marker hit by* $L \in \mathcal{L}$. $\square$

Corollary 3.6. *If* $\mathcal{L} \in \mathcal{EL}(S)$, *then for every* $\epsilon > 0$ *there exists an* ϵ*-marker hit by* $\mathcal{L}$. $\square$

The next lemma states that hitting a marker is an open condition.

Lemma 3.7. *If* $\mathcal{L} \subset \mathcal{L}(S)$ *hits the marker* $\mathcal{M}$, *then there exists a* $\delta > 0$ *such that if* $\mathcal{L}' \in \mathcal{L}(S)$*and* $\mathcal{L} \subset N_{PT(S)}(\mathcal{L}', \delta)$, *then* $\mathcal{L}'$ *hits* $\mathcal{M}$. $\square$

By super convergence we have

Corollary 3.8. *If* $x \in \mathcal{PML}(S)$ *is such that* $\phi(x)$ *hits the marker* $\mathcal{M}$, *then there exists an open set* U *containing* x *such that* $y \in U$ *implies that* $\phi(y)$ *hits* $\mathcal{M}$. $\square$

Lemma 3.9. *Let* $C \subset S$ *a simple closed geodesic. There exists a* $k > 0$ *such that if* $\mathcal{L} \in \mathcal{L}(S)$ *and* $|C \cap \mathcal{L}| > k$, *then there exists a* C*-marker that is hit by* $\mathcal{L}$.

Proof. An elementary topological argument shows that if k is sufficiently large, then there exists a leaf L containing 5 distinct, though possibly overlapping embedded subarcs $u_1, \cdots, u_5$ with endpoints in C which represent the same path homotopy class rel C such that the following holds. Each arc u_j intersects C more than $4g + p + 1$ times and fixing a preimage $\tilde{C}$ of C to $\tilde{S}$, these arcs have lifts to arcs $\bar{u}_1, \cdots, \bar{u}_5$ in $\tilde{S}$ starting at $\tilde{C}$ and ending at the same preimage $\tilde{C}'$. After reordering we can assume that $\bar{u}_1$ and $\bar{u}_5$ are outermost.

Let $\hat{\alpha}_0$ and $\hat{\alpha}_1$ be the maximal closed arcs respectively in $\tilde{C}$ and $\tilde{C}'$ with endpoints in $\cup\bar{u}_i$ and for $i = 0, 1$ let $\bar{\alpha}_i = \pi(\hat{\alpha}_i)$, where π is the universal covering map.

This gives rise to a C-marker $\mathcal{M}$ with posts α_0, α_1 where $\alpha_i = \mathrm{int}(\bar{\alpha}_i)$ and u_2, u_3, u_4 represent the path homotopy class. Note that if the geodesic arc β spans $\bar{\mathcal{M}}$, then β lifts to $\hat{\beta}$ with endpoints in $\hat{\alpha}_0$ and $\hat{\alpha}_1$. Being a geodesic it lies in the geodesic rectangle formed by $\hat{\alpha}_0, \hat{\alpha}_1, \bar{u}_1, \bar{u}_5$. Thus it intersects C more than $4g+p+1$ times. $\square$

In the rest of this section V will denote the underlying space of a finite simplicial complex. In application, $V = B^k$ or $S^k \times I$.

Definition 3.10. A *marker family* $\mathcal{J}$ of V is a finite collection $(\mathcal{M}_1, W_1), \cdots, (\mathcal{M}_m, W_m)$ where each $\mathcal{M}_i$ is a marker and each W_j is a compact subset of V. Let $f : V \to \mathcal{PML}\mathcal{EL}(S)$. We say that f *hits* the marker family $\mathcal{J}$ if for each $1 \le i \le m$ and $t \in W_i$, $\phi(f(t))$ hits $\mathcal{M}_i$. Let $C = \{b_1, \cdots, b_q\}$ be a set of simple closed geodesics. We say that $\mathcal{J}$ is f*-free of* C if for each $1 \le i \le m$ and $t \in W_i$, $\mathcal{M}_i$ is $\phi(f(t))$-free of C. More generally, if $U \subset V$, then we say that f *hits* $\mathcal{J}$ *along* U (resp. $\mathcal{J}$ *is* f*-free of* C *along* U) if for each $1 \le i \le m$ and $t \in U \cap W_i$, $\phi(f(t))$ hits $\mathcal{M}_i$ (resp.

$\mathcal{M}_i$ is $\phi(f(t))$-free of C). We say that the homotopy $F : V \times I \to \mathcal{PMLEL}(S)$ is $\mathcal{J}$-marker preserving if for each $t \in I$, $F|V \times t$ hits $\mathcal{J}$.

Note that if $\mathcal{J}$ is F-free of C, then F is in particular a $\mathcal{J}$-marker preserving homotopy.

An *ϵ-marker cover* (resp. *C-marker cover*) of V is a marker family $(\mathcal{M}_1, W_1), \cdots, (\mathcal{M}_m, W_m)$ where each $\mathcal{M}_i$ is an ϵ-marker (resp. C-marker) and the interior of the W_i's form an open cover of V.

The next lemma gives us conditions for constructing ϵ and C-marker families.

Lemma 3.11. *Let S is a finite type hyperbolic surface such that* $\dim(\mathcal{PML}(S)) = 2n+1$. *Let V be a finite simplical complex.*

i) If $\epsilon > 0$ and $f : V \to \mathcal{PMLEL}(S)$ is a generic PL map such that $\dim(V) \leq n$, *then there exists an ϵ-marker family $\mathcal{E}$ hit by f.*

ii) Given the simple closed geodesic C, there exists $N(C) \in \mathbb{N}$ such that if $f : V \to \mathcal{PMLEL}(S)$ is such that for all $t \in V$, $|\phi(f(t)) \cap C| \geq N(C)$, then there exists a C-marker family $\mathcal{S}$ hit by f.

Proof of i). Since $k \leq n$ and f is generic, for each $t \in V$, $\mathcal{A}(\phi(t)) \neq \emptyset$, where $\mathcal{A}(\phi(t))$ is the arational sublamination of $\phi(t)$. By Lemma 3.4 for each $t \in V$ there exists an ϵ-marker $\mathcal{M}_t$ and compact set W_t such that $t \in \text{int}(W_t)$ and for each $s \in W_t$, $\phi(f(s))$ hits $\mathcal{M}_t$. The result follows by compactness of V. $\square$

Proof of ii). Given C, choose $N(C)$ as in Lemma 3.9. Thus for each $t \in V$ there exists a C-marker $\mathcal{M}_t$ and compact set U_t such that $t \in \text{int}(U_t)$ and for each $s \in U_t$, $\phi(f(s))$ hits $\mathcal{M}_t$. The result follows by compactness of V. $\square$

4. CONVERGENCE LEMMAS

This section establishes various criteria to conclude that a sequence of ending laminations converges to a particular ending lamination or to show that two ending laminations are close in $\mathcal{EL}(S)$. We also show that markers give neighborhood bases of elements $\mathcal{L} \in \mathcal{EL}(S)$ and sets in $\mathcal{PML}(S)$ of the form $\phi^{-1}(\mathcal{L})$, where $\mathcal{L} \in \mathcal{EL}(S)$.

Lemma 4.1. *Let $\mu \in \mathcal{EL}(S)$ and*

$$W_\epsilon(\mu) = \{\mathcal{L} \in \mathcal{EL}(S) | d_{PT(S)}(\mathcal{L}, \mu') < \epsilon, \mu' \textit{ is a diagonal extension of } \mu\}.$$

Then $\mathcal{W}(\mu) = \{W_\epsilon(\mu) | \epsilon > 0\}$ is a neighborhood basis of $\mu \in \mathcal{EL}(S)$.

Proof. By definition of coarse Hausdorff topology, $W_\epsilon(\mu)$ is open in $\mathcal{EL}(S)$. Therefore if the lemma is false, then there exists a sequence $\mathcal{L}_1, \mathcal{L}_2, \cdots$ such that $\mathcal{L}_i \in W_{1/i}(\mu)$ all i and a $c > 0$ such that for all $i, \mathcal{L}_i \notin N_{PT(S)}(\mu', c)$ for all diagonal extensions μ' of μ. After passing to subsequence we can assume that $\{\mathcal{L}_i\} \to \mathcal{L}_\infty$ with respect to the Hausdorff topology. If $\mathcal{L}_\infty$ is not a diagonal extension of μ, then $\mathcal{L}_\infty$ is transverse to each diagonal extension of μ and hence there exists an $\epsilon > 0$ such that $d_{PT(S)}(\mu', \mathcal{L}_i) > \epsilon$ for all i sufficiently large and every diagonal extension of μ, a contradiction. $\square$

Lemma 4.2. *Let $\mu \in \mathcal{EL}(S)$, μ' a diagonal extension and $x_1, x_2, \cdots \in \mathcal{PML}(S)$ such that $\lim_{i\to\infty} d_{PT(S)}(\phi(x_i), \mu') = 0$, then after passing to subsequence $x_i \to x_\infty \in \phi^{-1}(\mu)$.*

Proof. After passing to subsequence we can assume that $x_i \to x_\infty \in \mathcal{PML}(S)$. If $\phi(x_\infty) \neq \mu$, then $\phi(x_\infty)$ intersects μ transversely. Let $p \in \phi(x_\infty) \cap \mu$ and L the leaf of μ containing p. Then $\phi(x_\infty)$ intersects L at p at some angle $\theta > 0$. By super convergence, for i sufficiently large $\phi(x_i)$ intersects μ at $p_i \in L$ at angle θ_i, where p_i is very close to p (distance measured intrinsically in L) and θ_i is very close to θ. Since every leaf of μ' is dense in μ, it follows that there exists $N > 0$ such that if J is a segment of a leaf of μ' at least length N and i is sufficiently large, then $J \cap \phi(x_i) \neq \emptyset$ with angle of intersection at some point at least $\theta/2$. Thus $d_{PT(S)}(\phi(x_i), \mu')$ must be uniformly bounded below, else some $\phi(x_i)$ would have a transverse self intersection. □

Lemma 4.3. *If $K \subset \mathcal{PML}(S)$ and $L \subset \mathcal{EL}(S)$ are compact and $K \cap \phi^{-1}(L) = \emptyset$, then there exists $\delta > 0$ such that if $d_{PT(S)}(\phi(x), \mu) < \delta$ where $x \in \mathcal{PML}(S)$ and $\mu \in L$, then $x \notin K$.*

Proof. Otherwise there exists sequences $x_1, x_2, \cdots \to x_\infty$, $\mu_1, \mu_2, \cdots \to \mu_\infty$ such that the $x_i's \in K$ and the $\mu_j's \in L$ and $lim_{i\to\infty} d_{PT(S)}(\phi(x_i), \mu_i) = 0$. After passing to subsequence we can assume that the μ_i's converge to a diagonal extension of μ_∞. This contradicts Lemma 4.2. □

Lemma 4.4. *If $L \subset \mathcal{EL}(S)$ is compact and $U \subset \mathcal{PML}(S)$ is open such that $\phi^{-1}(L) \subset U$, then there exists a neighborhood V of L such that $\phi^{-1}(V) \subset U$.*

Proof. Let $W_\epsilon(L) = \{\mathcal{L} \in \mathcal{EL}(S) | d_{PT(S)}(\mathcal{L}, \mu') < \epsilon, \mu' \text{is a diagonal extension of } \mu \in L\}$. Then $W_\epsilon(L)$ is open and for ϵ sufficiently small $\phi^{-1}(W_\epsilon(L)) \subset U$ otherwise taking $K = \mathcal{PML}(S) \setminus U$ one obtains a contradiction to the previous lemma. □

Lemma 4.5. *Let $\mu \in \mathcal{EL}(S)$ and $\mathcal{M}_1, \mathcal{M}_2, \cdots$ a sequence of markers such that for every $i \in \mathbb{Z}$, $\mathcal{M}_i$ is a $1/i$-marker hit by μ. If $U_i = \{\mathcal{L} \in \mathcal{EL}(S) | \mathcal{L}$ hits $\mathcal{M}_i\}$, then $\mathcal{U} = \{U_i\}$ is a neighborhood basis of μ in $\mathcal{EL}(S)$.*

Proof. By definition of $1/i$-marker, if $\mathcal{L}$ hits $\mathcal{M}_i$, then $d_{PT(S)}(\mathcal{L}, \mu) < 1/i$. Therefore, for all i, $U_i \subset W_{1/i}$. Since each U_i is open in $\mathcal{EL}(S)$ the result follows. □

Lemma 4.6. *Let $\mu \in \mathcal{EL}(S)$. For each $\epsilon > 0$ there exists $\delta > 0$ such that if $\{\mathcal{L}^1, \mathcal{L}^2, \cdots, \mathcal{L}^k, z\} \subset \mathcal{L}(S)$, $d_{PT(S)}(\mathcal{L}^k, z) < \delta$, $d_{PT(S)}(\mathcal{L}^i, \mathcal{L}^{i+1}) < \delta$ for $1 \leq i \leq k-1$ and $d_{PT(S)}(\mu', \mathcal{L}^1) < \delta$ for some diagonal extension μ' of μ, then $d_{PT(S)}(\mu, z) < \epsilon$.*

Proof. We give the proof for $k = 1$, the general case being similar. If the Lemma is false, then there exists a sequence $(\mathcal{L}_i, z_i, \delta_i)$ for which the lemma fails, where $\delta_i \to 0$. After passing to subsequence we can assume that $\mathcal{L}_i \to \mathcal{L}_\infty$ and $z_i \to z_\infty$ with respect to the Hausdorff topology. Since z_∞ is nowhere tangent to μ, it is transverse to μ. Since the ends of every leaf of every diagonal extension of μ is dense in μ and μ is filling, there exists $K > 0$ such that any length K immersed segment lying in a leaf of z_∞ intersects any length K segment lying in any diagonal extension of μ at some angle uniformly bounded away from 0. Thus a similar statement holds for each z_i, i sufficiently large, where K is replaced by $K + 1$. Therefore if δ_i is sufficiently small and i is sufficiently large, then $\mathcal{L}_i$ has length $K + 2$ immersed segments σ_1, σ_2 such that σ_1 is nearly parallel to a leaf of μ' and σ_2 is nearly parallel to a leaf of z_i. This implies that σ_1 nontrivially intersects σ_2 transversely, a contradiction. □

Lemma 4.7. *Let $\epsilon > 0$. Let $\tau_1, \tau_2, \cdots$ be a full unzipping sequence of the transversely recurrent train track τ_1. If each τ_i fully carries the geodesic lamination $\mathcal{L}$, then there exists $N > 0$ such that if $\mathcal{L}_1$ is carried by τ_i, for some $i \geq N$, then $d_{PT(S)}(\mathcal{L}_1, \mathcal{L}) < \epsilon$.*

Proof. This follows from the proof of Lemma 1.7.9 [PH] (see also Proposition 1.9 [G1]). That argument shows that each biinfinite train path of each τ_i is a uniform quasi-geodesic and that given $L > 0$, there exists $N > 0$ such that any length L segment lying in a leaf of a lamination carried by $\tau_i, i \geq N$, is isotopic to a leaf of $\mathcal{L}$ by an isotopy such that the track of a point has uniformly bounded length. $\square$

Lemma 4.8. *If τ is a train track that carries $\mu \in \mathcal{EL}(S)$, then τ fully carries a diagonal extension of μ.*

Proof. By analyzing the restriction of τ to each closed complementary region of μ, it is routine to add diagonals to μ to obtain a lamination fully carried by τ. $\square$

Lemma 4.9. *Let κ be a transversely recurrent train track that carries $\mu \in \mathcal{EL}(S)$. Given $\delta > 0$ there exists $N > 0$ so that if τ is obtained from κ by a sequence of $\geq N$ full splittings (i.e. along all the large branches) and τ carries both μ and $\mathcal{L} \in \mathcal{L}(S)$, then $d_{PT(S)}(\mathcal{L}, \mu) < \delta$.*

Proof. It suffices to show that $d_{PT(S)}(\mathcal{L}, \mu') < \delta_1$ for some diagonal extension μ' of μ and some $\delta_1 > 0$, that depends on δ and μ. Since there are only finitely many train tracks obtained from a given finite number of full splittings of κ it follows that if the lemma is false, then there exist $\tau_1, \tau_2, \cdots$ such that $\tau_1 = \kappa, \tau_i$ is a full splitting of τ_{i-1} and for each $i \in \mathbb{N}$ there exists $\mathcal{L}_i \in \mathcal{L}(S)$ carried by some splitting of τ_{n_i} with $d_{PT(S)}(\mathcal{L}_i, \mu) > \delta$ and $n_i \to \infty$. Note that $\mathcal{L}_i$ is also carried by τ_{n_i}. Since μ has only finitely many diagonal extensions we can assume from the previous lemma that each τ_i fully carries a fixed diagonal extension μ' of μ.

On the other hand, there is a full unzipping sequence $\tau'_1 = \tau_1, \tau'_2, \cdots$ with the property that each τ'_i carries exactly the same laminations as some τ_{m_i} and $m_i \to \infty$. Thus by Lemma 4.7 it follows that for i sufficiently large $d_{PT(S)}(\mathcal{L}_i, \mu') < \delta_1$, a contradiction. $\square$

For once and for all fix a parametrized pants decomposition of S, with the corresponding finite set of standard train tracks.

Proposition 4.10. *Given $\epsilon > 0, \mu \in \mathcal{EL}(S)$, there exists $N > 0, \delta > 0$ such that if τ is obtained from a standard train track by N full splittings and τ carries μ, then the following holds. If $\mathcal{L} \in \mathcal{L}(S)$ is carried by τ, $z \in \mathcal{L}(S)$ and $d_{PT(S)}(\mathcal{L}, z) < \delta$, then $d_{PT(S)}(\mu, z) < \epsilon$. In particular, if $\mathcal{L}$ is carried by τ, then $d_{PT(S)}(\mu, \mathcal{L}) < \epsilon$.*

Proof. Apply Lemmas 4.9 and 4.6. $\square$

5. A criterion for constructing continuous maps of compact manifolds into $\mathcal{EL}(S)$

This section is a generalization of the corresponding one of [G1] where a criterion was established for constructing continuous paths in $\mathcal{EL}(S)$. Our main result is much more general and is technically much simpler to verify. It will give a criterion for extending a continuous map $g : S^{k-1} \to \mathcal{EL}(S)$ to a continuous map $\mathcal{L} : B^k \to \mathcal{EL}(S)$, though it is stated in a somewhat more general form.

Recall that our compact surface S is endowed with a fixed hyperbolic metric. Let $\{C_i\}_{i\in\mathbb{N}}$ denote the set of simple closed geodesics in S.

Notation 5.1. If $\mathcal{U}_j$ is a finite open cover of a compact set V, then its elements will be denoted by $U_j(1), \cdots U_j(k_j)$.

Proposition 5.2. *Let V be the underlying space of a finite simplicial complex and W the subspace of a subcomplex. Let $g : W \to \mathcal{EL}(S)$ and for $i \in \mathbb{N}$ let $f_i : V \to \mathcal{PMLEL}(S)$ be continuous extensions of g. Let $\mathcal{L}_m(t)$ denote $\phi(f_m(t))$. Let $\epsilon_1, \epsilon_2, \cdots$ be such that for all i, $\epsilon_i/2 > \epsilon_{i+1} > 0$. Let $\mathcal{U}_1, \mathcal{U}_2, \cdots$ be a sequence of finite open covers of V. Suppose that each $U_j(k)$ is assigned both an ϵ_j-marker $\alpha_j(k)$ and a C_j-marker $\beta_j(k)$. Assume that the following two conditions hold.*

(sublimit) For each $t \in U_j(k)$ and $m \geq j$, $\mathcal{L}_m(t)$ hits $\alpha_j(k)$.

(filling) For each $t \in U_j(k)$ and $m \geq j$, $\mathcal{L}_m(t)$ hits $\beta_j(k)$.

Then there exists a continuous map $\mathcal{L} : V \to \mathcal{EL}(S)$ extending g so that for $t \in V$, $\mathcal{L}(t)$ is the coarse Hausdorff limit of $\{\mathcal{L}_m(t)\}_{m\in\mathbb{N}}$.

Proof. Fix t. We first construct a minimal and filling $\mathcal{L}(t)$. After passing to subsequence we can assume that the sequence $\mathcal{L}_{m_i}(t)$ converges in the Hausdorff topology to a lamination $\mathcal{L}'(t)$. If $t \in U_i(j)$, then the filling and sublimit conditions imply that if $k > i$, then some arcs $\gamma_i(k), \sigma_i(k)$ in leaves of $\mathcal{L}_k(t)$ respectively span the ϵ_i and C_i-markers $\alpha_i(j)$ and $\beta_i(j)$. This implies that arcs in $\mathcal{L}'(t)$ span the corresponding closed markers and hence, $\mathcal{L}'(t)$ intersects each C_i transversely and hence $\mathcal{L}'(t)$ contains no closed leaves. Thus spanning arcs in $\mathcal{L}'(t)$ are embedded (as opposed to wrapping around a closed geodesic) and hence $|\mathcal{L}'(t) \cap C_i| > 4g + p + 1$ for all i. Let $\mathcal{L}(t)$ be a minimal sublamination of $\mathcal{L}'(t)$. If $\mathcal{L}(t)$ is not filling, then there exists a simple closed geodesic C, disjoint from $\mathcal{L}(t)$ that can be isotoped into any neighborhood of $\mathcal{L}(t)$ in S. An elementary topological argument shows that $|C \cap \mathcal{L}'(t)| \leq 4g + p + 1$, contradicting the filling condition.

We next show that $\mathcal{L}(t)$ is independent of subsequence. Let $\mathcal{L}'_0(t) \in \mathcal{EL}(S)$ be a lamination that is the Hausdorff limit of the subsequence $\{\mathcal{L}_{k_i}(t)\}$ and $\mathcal{L}_0(t)$ the sublamination of $\mathcal{L}'_0(t)$ in $\mathcal{EL}(S)$. By the sublimit condition each of $\mathcal{L}'(t), \mathcal{L}'_0(t)$ have arcs that span the same set $\{\bar{\alpha}_i\}$ of closed markers, where α_i is an ϵ_i-marker with associated open set $U_i \subset V$, where $t \in U_i$. Since $\epsilon_i \to 0$, the lengths of the initial posts $\{\alpha_{i_0}\}$ go to 0. Thus after passing to a subsequence of the initial posts, $\{\alpha_{i_{j_0}}\} \to x \in S$. Now let v_{i_j} be the unit tangent vector to the initial point of some spanning arc of α_{i_j}. After passing to another subsequence, $v_{i_j} \to v$ a unit tangent vector to x. The sublimit condition implies that v is tangent to a leaf of both $\mathcal{L}'(t)$ and $\mathcal{L}'_0(t)$ and hence $\mathcal{L}'(t)$ and $\mathcal{L}'_0(t)$ have a leaf in common. It follows that $\mathcal{L}(t) = \mathcal{L}_0(t)$.

We apply Lemma 1.9 to show that f is continuous at t. Let v and $\{\alpha_{i_j}\}_{i\in\mathbb{N}}$ be as in the previous paragraph, where $\{\alpha_{i_j}\}_{i\in\mathbb{N}}$ is the final subsequence produced in that paragraph. Fix $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that for $i \geq N$, $d_{PT(S)}(v'_{i_j}, v) \leq \epsilon$ where v'_{i_j} is any unit tangent vector to the initial point of a spanning arc of $\bar{\alpha}_{i_j}$. Therefore if $m \geq N_j$ and $s \in U_{N_j}$, then $d_{PT(S)}(\mathcal{L}_m(s), v) \leq \epsilon$. Since this is true for all $m \geq N_j$ it follows that for all $s \in U_{N_j}$, $d_{PT(S)}(\mathcal{L}'(s), \mathcal{L}'(t)) \leq \epsilon$. $\square$

6. Pushing off of B_C

Given a generic PL map $f : B^k \to \mathcal{PMLEL}(S)$ and a simple closed geodesic C, this section will describe homotopies of f such that if f_1 is a resulting map, then $f_1^{-1}(B_C) = \emptyset$. The map f_1 is said to be obtained from f by *pushing off of* C. Various technical properties associated with such push off's will be obtained. The concept of *relatively pushing* f *off of* C will be introduced and analogous technical results will be established. In subsequent sections we will produce a sequence $f_1, f_2, \cdots$ satisfying the hypothesis of Proposition 5.2, where f_{i+1} is obtained by relatively pushing f_i off of a finite set of geodesics, one at a time.

Remark 6.1. Recall the convention that n is chosen so that $\dim(\mathcal{PML}(S)) = 2n+1$. Let C be a simple closed geodesic. Let λ_C denote the projective measure lamination with support C. As in [G1], we denote by B_C the PL $2n$-ball consisting of those projective measured laminations that have intersection number 0 with λ_C. Recall that B_C is the cone of the PL $(2n-1)$-sphere δB_C to λ_C, where δB_C consists of those points of B_C which do not have C as a leaf. Furthermore, if $x \in B_C \setminus \lambda_C$, then $x = p((1-t)\hat{\lambda} + t\hat{\lambda}_C)$, for some $\hat{\lambda} \in \mathcal{ML}(S)$ representing a unique $\lambda \in \delta C$, some $\hat{\lambda}_C \in \mathcal{ML}(S)$ representing λ_C and some $t < 1$.

Definition 6.2. The *ray through* $x \in B_C \setminus \lambda_C$ is the set of points $r(x)$ in $\mathcal{PML}(S)$ represented by measured laminations of the form $\{t\hat{\lambda} + (1-t)\hat{\lambda}_C | 0 \le t \le 1\}$ where $\hat{\lambda}$ and $\hat{\lambda}_C$ are as above. If $K \subset B_C$ and $K \cap \lambda_C = \emptyset$, then define $r(K) = \cup_{x \in K} r(x)$.

Remarks 6.3. Note that $r(x)$ is well defined and $r(K)$ is compact if K is compact.

Using the methods of [Th1], [PH] or [G1] it is routine to show that there exists a neighborhood of B_C homeomorphic to $2B^{2n} \times [-1,1]$, where $2B^{2n}$ denotes the radius-2 $2n$-ball about the origin in $\mathbb{R}^{2n}$, such that B_C is identified with $B^{2n} \times 0$, λ_C is identified with $(0,0)$ and for each $x \in B_C \setminus \lambda_C$, $r(x)$ is identified with a ray through the origin with an endpoint on $S^{2n-1} \times 0$.

While the results in this section are stated in some generality, on first reading one should imagine that if $f : V \to \mathcal{PMLEL}(S)$, then $V = B^k$ and $f^{-1}(\mathcal{EL}(S)) = S^{k-1}$, where $k \le 2n$.

Definition 6.4. Let V be the underlying space of a finite simplicial complex and W that of a subcomplex. If $f : V \to \mathcal{PMLEL}(S)$ and $W = f^{-1}(\mathcal{EL}(S))$, then the generic PL map $f_1 : V \to \mathcal{PMLEL}(S)$ is said to be obtained from f by δ*-pushing off of* B_C if there exists a homotopy $F : V \times I \to \mathcal{PMLEL}(S)$, called a (C,δ) *push off homotopy* such that

i) $f_1^{-1}(B_C) = \emptyset$,

ii) $F(t,s) = f(t)$ if either $s = 0$ or $d_V(t, f^{-1}(B_C)) \ge \delta$ or $d_V(t, W) \le \delta$ or $d_{\mathcal{PML}(S)}(f(t), B_C)) \ge \delta$ and

iii) for each $t \in V$ such that $d_{\mathcal{PML}(S)}(f(t), B_C) < \delta$ there exists an $x \in f(V) \cap B_C$ such that for all $s \in [0,1]$, $d_{\mathcal{PML}(S)}(F(t,s), r(x)) < \delta$. Furthermore if $F(t,s) \in B_C$, then $F(t,s) \in r(f(t))$.

Lemma 6.5. *If $f : V \to \mathcal{PMLEL}(S)$ is a generic PL map, $\dim(V) \le 2n$ and C is a simple closed geodesic, then for all sufficiently small $\delta > 0$ there exists a (C,δ) push off homotopy of f.*

Proof. It follows by super convergence and compactness that if C is a simple closed geodesic, then $f^{-1}(B_C)$ is a compact set disjoint from some neighborhood of W. By genericity of f, i.e. Lemma 2.4, there exists an $\epsilon_1 > 0$ such that $d_{\mathcal{PML}(S)}(f(V), \lambda_C) \geq \epsilon_1$. Consider a natural homotopy $F_\epsilon : ((2B^{2n} \setminus \epsilon B^{2n}) \times [-1,1]) \times I \to 2B^{2n} \times [-1,1]$ from the inclusion to a map whose image is disjoint from $B^{2n} \times 0$, which is supported in an ϵ-neighborhood of $(B^{2n} \setminus \epsilon B^{2n}) \times 0$ and where points in $(B^{2n} \setminus \epsilon B^{2n}) \times 0$ are pushed radially out from the origin. Let $g : N(B_C) \to 2B^{2n} \times [-1,1]$ denote the parametrization given by Remark 6.3. The desired (C, δ)-homotopy is obtained by appropriately interpolating the trivial homotopy outside of a very small neighborhood of $f^{-1}(B_C)$ with $g^{-1} \circ F_\epsilon \circ g \circ f$ restricted to a small neighborhood of $f^{-1}(B_C)$ where ϵ is sufficiently small and then doing a small perturbation to make f_1 generic. □

Lemma 6.6. *Let C be a simple closed geodesic and $f : V \to \mathcal{PML}\mathcal{EL}(S)$ be a generic PL map with* $\dim(V) \leq 2n$. *Let $\mathcal{J}$ be a marker family of V hit by f that is free of C. If δ is sufficiently small, then any (C, δ) homotopy F from f to f_1 is $\mathcal{J}$ marker preserving, free of C.*

Proof. Since there are only finitely many markers in a marker family, it suffices to show that if $K \subset V$ is compact and $\phi(f(t))$ hits the marker $\mathcal{M}$ free of C at all $t \in K$, then for δ sufficiently small $\phi(F(t,s))$ hits $\mathcal{M}$ free of C at all $t \in K$ and $s \in I$. This is a consequence of super convergence and compactness. Indeed, if $x \in f(K) \cap B_C$, then there exists a leaf of $\phi(x)$ distinct from C hits $\mathcal{M}$. Since all points in $r(x) \setminus \lambda_C$ have the same underlying lamination this fact holds for all $y \in r(x)$. By super convergence it holds at all points in a neighborhood of $r(x)$ in $\mathcal{PML}(S)$. Let U be the union of these neighborhoods over all $x \in f(K) \cap B_C$. By compactness of K and B_C, there exists a $\eta > 0$ such that if $y \in B_C$ and $d_{\mathcal{PML}(S)}(y, f(t)) \leq \eta$ for some $t \in K$, then $N_{\mathcal{PML}(S)}(r(y), \eta) \subset U$. Any (C, δ) homotopy with $\delta < \eta$ satisfies the conclusion of the lemma. □

Definition 6.7. If $x \in \mathcal{PML}(S)$ and A is a simple closed geodesic, then define $g(x, A) \in \mathbb{Z}_{\geq 0} \cup \infty$ the *geometric intersection number* of x with A by $g(x, A) = \min\{|\phi(x) \cap A'| | A'$ is isotopic to $A\}$. If $f : V \to \mathcal{PML}\mathcal{EL}(S)$ define the *geometric intersection number of f with A* by $g(f, A) = \min\{g(\phi(f(t)), A) | t \in V\}$. If $0 < g(f, A) < \infty$, then we say that the multi-geodesic J is a *stryker curve for A* if for some $t \in V$, $J \subset \phi(f(t))$ and $|J \cap A| = g(f, A)$. We call J the *$f(t)$-stryker curve* or sometimes the *stryker curve at $f(t)$*.

Remark 6.8. Note that $|\phi(x) \cap A'|$ is minimized when $A = A'$ unless A is a leaf of $\phi(x)$ in which case $g(x, A) = 0$.

Lemma 6.9. *If $f : V \to \mathcal{PML}\mathcal{EL}(S), A$ is a simple closed geodesic and $0 < g(f, A) < \infty$, then the set of stryker curves is finite. Also $m(f, A) = \{t \in V | |\phi(f(t)) \cap A| = g(f, A)\}$ is compact. Finally $m(f, A)$ is the disjoint union of the compact sets $m_{J_1}(f, A), \cdots, m_{J_m}(f, A)$ where $t \in m_{J_i}(f, A)$ implies that J_i is the stryker curve at $f(t)$.*

Proof. Super convergence implies that $V \setminus m(f, A)$ is open, hence $m(f, A)$ is compact. If the first assertion is false, then there exists $t_1, t_2, \cdots$ converging to t such that if J_i denotes the stryker curve at t_i, then the J_i's are distinct. By compactness, $t \in m(f, A)$, so let J be the stryker curve at t. Super convergence implies that if s is sufficiently close to t, then either $s \notin m(f, A)$ or J is the stryker curve at s, a contradiction. The final assertion again follows from super convergence. □

Lemma 6.10. *Let $f : V \to \mathcal{PMLEL}(S)$ be a generic PL map and A and C disjoint simple closed geodesics such that $0 < g(f, A) < \infty$. Then for δ sufficiently small, any (C, δ) push off f_1 satisfies $g(f_1, A) \geq g(f, A)$. If equality holds and J is a stryker curve for f_1, then J is a stryker curve for f.*

Proof. If $t \in f^{-1}(B_C)$, then $|(\phi(f(t)) \setminus C) \cap A| = |\phi(f(t)) \cap A| \geq g(f, A)$. By super convergence there exists a neighborhood U' of $r(f(t))$ such that $y \in U'$ implies that $g(\phi(y), A) \geq g(f, A)$ and hence there exists a neighborhood U of $r(f(V) \cap B_C)$ with the same property. If δ is sufficiently small to have any (C, δ) push off homotopy supported in U, then $g(f_1, A) \geq g(f, A)$.

Now assume that equality holds. If $t \in m(f, A) \cap f^{-1}(B_C)$ and J is the stryker curve at $f(t)$, then by super convergence there exists a neighborhood U' of $r(f(t))$ such that if $x \in U'$ and $g(x, A) = g(f, A)$, then J is the stryker curve at x. Thus, there exists a neighborhood U of $f(V) \cap B_C$ such that $x \in U$ and $g(x, A) = g(f, A)$, then the stryker curve at x is a stryker curve of f. If δ is sufficiently small to have any (C, δ) push off of f supported in U, then the second conclusion holds. □

This argument proves the following sharper result.

Lemma 6.11. *If $f : V \to \mathcal{PMLEL}(S)$ and $\eta > 0$, then there exists $\delta > 0$ such that if δ is sufficiently small and f_1 is the result of a (C, δ) push off homotopy of f, then $m(f_1, A) \subset N_V(m(f, A), \eta)$ and if $t \in m(f_1, A)$ and $d_V(t, m_{J_i}(f, A)) < \eta$, then J_i is the stryker curve at $f_1(t)$.* □

We need *relative versions* of generalizations of the above results.

Definition 6.12. Let V be the underlying space of a finite simplicial complex and W that of a finite subcomplex. Let $f : V \to \mathcal{PMLEL}(S)$ be a generic PL map with $f^{-1}(\mathcal{EL}(S)) = W$. Let $K \subset f^{-1}(B_C)$ be closed. We say that the generic PL map $f_1 : V \to \mathcal{PMLEL}(S)$ is obtained from f by (C, δ, K) *pushing off* if there exists a homotopy $F : V \times I \to \mathcal{PMLEL}(S)$ called a (C, δ, K) *push off homotopy* such that

i) $f_1(K) \cap B_C = \emptyset$,

ii) $F(t, s) = f(t)$ if either $s = 0$ or $d_V(t, K) \geq \delta$ or $d_V(t, W) \leq \delta$ or $d_{\mathcal{PML}(S)}(f(t), f(K)) \geq \delta$.

iii) for each $t \in V$ such that $d_{\mathcal{PML}(S)}(f(t), f(K)) < \delta$ there exists an $x \in r(f(K))$ such that for all $s \in [0, 1]$, $d_{\mathcal{PML}(S)}(F(t, s), r(x)) < \delta$. Furthermore, if $F(t, s) \in B_C$, then $F(t, s) \in r(f(t))$ and if $f_1(t) \in \mathrm{int}(B_C)$, then $f(t) \in \mathrm{int}(B_C)$.

Lemma 6.13. *If $f : V \to \mathcal{PMLEL}(S)$ is a generic PL map, $\dim(V) \leq 2n$, C a simple closed geodesic and K a closed subset of $f^{-1}(B_C)$, then for every sufficiently small $\delta > 0$ there exists a (C, δ, K) push off homotopy of f.*

Proof. For δ sufficiently small let $U \subset V \setminus N_V(W, \delta)$ be open such that $K \subset U \subset N_V(K, \delta/10) \cap f^{-1}(N_{\mathcal{PML}(S)}(r(f(K)), \delta/10))$. Let $\rho : V \to [0, 1]$ be a continuous function such that $\rho(K) = 1$ and $\rho(V \setminus U) = 0$. If $F(t, s)$ defines a (C, δ) push off homotopy, then $F(t, \rho(t)s)$ suitably perturbed defines a (C, δ, K) push off homotopy. □

Lemma 6.14. *Let V be a finite p-complex and $f : V \to \mathcal{PMLEL}(S)$ a generic PL map, C a simple closed geodesic, $L \subset \mathcal{PML}(S)$ a finite q-subcomplex of $\mathcal{C}(S)$ and K a closed subset of $f^{-1}(B_C)$. If $p+q \leq 2n-1$ or $p \leq n$, then for every sufficiently small $\delta > 0$ any (C, δ, K) push off homotopy of f is supported away from L.*

Proof. Let $Z = (\partial B_C \cap L) * C$. By Lemma 2.4, $f(V) \cap (Z \cup L) = \emptyset$ and hence $r(f(K)) \cap (Z \cup L) = \emptyset$. Thus the conclusion of the lemma holds provided $\delta < d_{\mathcal{PML}(S)}(r(f(K)), Z \cup L)$. $\square$

We have the following relative version of Lemma 6.6.

Lemma 6.15. *Let $b_1, b_2, \cdots, b_r, C$ be simple closed geodesics, $f : V \to \mathcal{PMLEL}(S)$ a generic PL map and $K \subset f^{-1}(B_C)$ be compact. Let $\mathcal{J}$ be a marker family that is f-free of $\{b_1, \cdots, b_r, C\}$. If δ is sufficiently small, then $\mathcal{J}$ is F-free of $\{b_1, \cdots, b_r, C\}$ for any (C, δ, K) push off homotopy F of f.* $\square$

Definition 6.16. Let $f : V \to \mathcal{PMLEL}(S)$. Let Y be a compact subset of V and A a simple closed geodesic. Define $g(f, A; Y) = \min\{g(f(t), A) | t \in Y\}$, the *$Y$-geometric intersection number of f and A*. If $0 < g(f, A; Y) < \infty$, define the *Y-stryker curves for f and A* to be those multi-geodesics σ such that for some $t \in Y$, $\sigma \subset \phi(f(t))$ and $|\sigma \cap A| = g(f, A, Y)$. Define $m(f, A; Y) = \{t \in Y | g(f(t), A) = g(f, A; Y)\}$ and if σ is a Y-stryker curve then define $m_\sigma(f, A; Y) = \{t \in Y | \sigma \subset \phi(f(t))$ and $|\sigma \cap A| = g(f, A, Y)\}$. We say that the set B of simple closed geodesics *solely hits the marker $\mathcal{M}$ at t* if each leaf of the lamination $\phi(f(t))$ that hits $\mathcal{M}$ lies in B. If $Z \subset V$ then let $S(f, \mathcal{M}, B, Z)$ denote the set of points in Z where f solely hits $\mathcal{M}$.

The proof of Lemma 6.9 holds for in the relative case.

Lemma 6.17. *If $f : V \to \mathcal{PMLEL}(S)$, A is a simple closed geodesic, $Y \subset V$ is compact and $0 < g(f, A; Y)$, then the set of Y-stryker curves is finite. Also $m(f, A; Y)$ is compact and is the disjoint union of compact sets $m_{J_1}(f, A; Y), \cdots, m_{J_m}(f, A; Y)$ where $t \in m_{J_i}(f, A; Y)$ implies that J_i is a Y-stryker curve at $f(t)$.* $\square$

By super convergence we have the following result.

Lemma 6.18. *If $f{:}V \to \mathcal{PMLEL}(S)$ is a generic PL map, B is a finite set of simple closed geodesics, $Z \subset V$ is compact, then $S(f, \mathcal{M}, B, Z)$ is compact.* $\square$

We have the following analogy of Lemma 6.10.

Lemma 6.19. *Let $f : V \to \mathcal{PMLEL}(S)$, be a generic PL map, $\eta > 0$ and A and C disjoint simple closed geodesics. Let K be a closed subset of $f^{-1}(B_C)$ and $Y \subset V$ be compact. If $0 < g(f, A; Y) < \infty$, then there exists a neighborhood U of Y such that for δ sufficiently small, any (C, δ, K) push off f_1 satisfies $g(f_1, A; \bar{U}) \geq g(f, A; Y)$. If equality holds, then $m(f_1, A; \bar{U}) \subset N_V(m(f, A; Y), \eta)$ and if $t \in m(f_1, A; \bar{U})$ and $d_V(t, m_{J_i}(f, A; Y)) < \eta$, then J_i is the $\bar{U}$-stryker curve to A at $f_1(t)$.* $\square$

7. Marker tags

Definition 7.1. Let A be a simple closed multi-geodesic in S. We say that τ is a *tag* for A, if τ is a compact embedded geodesic curve (with $\partial\tau$ possibly empty) transverse to A such that $\partial\tau \subset A$ and $\mathrm{int}(\tau) \cap A \neq \emptyset$.

Let $\mathcal{M}$ be a marker hit by the simple closed multi-geodesic A. Then $r \geq 3$ distinct subarcs of A span $\mathcal{M}$, where $r \in \mathbb{N}$ is maximal. These arcs run from α_0 to α_1, the posts of $\mathcal{M}$. Suppose that the initial points of these arcs intersect α_0 at $c_1, \cdots, c_r$. Let τ be the maximal subarc of α_0 with endpoints in $\{c_1, \cdots, c_r\}$. Such a tag is called a *marker tag*.

Given $f : V \to \mathcal{PMLEL}(S)$ that hits the marker $\mathcal{M}$ we may need to find a new f that hits the marker $\mathcal{M}$ free of a particular multi-geodesic. Tags are introduced to measure progress in that effort. We will find a sequence of push off homotopies whose resulting maps intersect a given tag more and more so that we can ultimately invoke the following result.

Lemma 7.2. *Let $f : V \to \mathcal{PMLEL}(S)$, $\mathcal{M}$ a marker, A a simple closed multi-geodesic that hits $\mathcal{M}$ and τ the corresponding marker tag. Let $b_1, \cdots, b_r$ be simple closed geodesics such that for all $t \in f^{-1}(B_A)$, $|(\phi(f(t)) \setminus (\cup_{i=1}^r b_i \cup A))| \cap \tau \geq 3(3g-3+p)$, then $\mathcal{M}$ is f-free of $\{A, b_1, \cdots, b_r\}$ along $f^{-1}(B_A)$.*

Proof. If $f(t) \in B_A$, then any leaf L of $\phi(f(t))$ distinct from A with $L \cap \tau \geq m$ has at least m distinct subarcs that span $\mathcal{M}$. If L is a non compact leaf of $\phi(f(t))$ and $L \cap \tau \neq \emptyset$, then $|L \cap \tau| = \infty$, since L is non proper. If only closed geodesics of $\phi(f(t))$ intersect τ, then since $\phi(f(t))$ can have at most $3g-3+p$ such geodesics, one of them L distinct from $\{b_1, \cdots, b_r, A\}$ must satisfy $|L \cap \tau| \geq 3$. □

Definition 7.3. Let τ be a tag for the multi-geodesic A, $f : V \to \mathcal{PMLEL}(S)$ a generic PL map and Y a compact subset of $f^{-1}(B_A)$. Define $g(f, \tau, Y) = \min\{|(\phi(f(t)) \setminus A) \cap \tau||t \in Y\}$ the *geometric intersection number of f with τ along Y*.

If $0 < g(f, \tau, Y) < \infty$, then define the multi-geodesic J to be a *Y-stryker curve* for τ if $J \subset \phi(f(t))$, $J \cap A = \emptyset$ and $|J \cap \tau| = g(f, \tau, Y)$.

The proof of Lemma 6.9 readily generalizes to the following result.

Lemma 7.4. *If $f : V \to \mathcal{PMLEL}(S)$ is a generic PL map, τ is a tag for the simple closed geodesic A and Y is closed in $f^{-1}(B_A)$, then the set of Y-stryker curves is finite. Also the set $m(f, \tau, Y) = \{t \in Y | g(f, \tau, Y) = |(\phi(f(t)) \setminus A) \cap \tau|\}$ is compact and and canonically partitions as the disjoint union of the compact sets $m_{J_1}(f, \tau, Y), \cdots, m_{J_k}(f, \tau, Y)$ where J_i is the Y-stryker curve to τ at all $t \in m_{J_i}(f, \tau, Y)$.* □

Similarly, Lemma 6.19 generalizes to the following result.

Lemma 7.5. *Let $f : V \to \mathcal{PMLEL}(S)$ be a generic PL map, τ a tag for the simple closed geodesic A, Y a closed subset of $f^{-1}(B_A)$ and $0 < g(f, \tau, Y) < \infty$. Let C be a simple closed geodesic such that $C \cap (A \cup \tau) = \emptyset$ and K a closed subset of $f^{-1}(B_C)$. If $\eta > 0$, then there exists a neighborhood U of Y such that for δ sufficiently small, any (C, δ, K) push off f_1 satisfies $g(f_1, \tau, Y_1) \geq g(f, \tau, Y)$, where $Y_1 = f_1^{-1}(B_A) \cap \bar{U}$. If equality holds, then $m(f_1, \tau, Y_1) \subset N_V(m(f, \tau, Y), \eta)$ and if $t \in m(f_1, \tau, Y_1)$ and $d_V(t, m_{J_i}(f, \tau, Y)) < \eta$, then J_i is the Y_1 stryker curve to τ at $f_1(t)$. In particular if J is a Y_1-stryker curve for f_1 and τ, then J is a Y-stryker curve for f and τ.* □

8. Marker Cascades

This very technical section begins to address the following issue. To invoke Proposition 5.2 we need to find a sequence $f_1, f_2, \cdots$ satisfying the sublimit and filling conditions, in particular satisfying the property that $f_j^{-1}(B_{C_i}) = \emptyset$ for $i \leq j$. We cannot just create f_i from f_{i-1} by pushing off of C_i, because C_i may be needed to hit previously constructed markers. To make C_i free of these markers we may need to relatively push off of other curves. We may not be able to push off of those curves because they in turn are needed to hit markers. In subsequent sections we shall see that finiteness of S, genericity of f and the $k \le n$ condition will force this process to terminate. Thus before we push off of C_i we will do a sequence of relative push offs of other curves.

We introduce the notion of *marker cascade* to keep track of progress. Given $f : V \to \mathcal{PMLEL}(S)$, markers $\mathcal{M}_1, \cdots, \mathcal{M}_m$ and pairwise disjoint simple closed curves $a_1, \cdots, a_v$ a marker cascade is a (complicated) measure of how far $\mathcal{M}_1, \cdots, \mathcal{M}_m$ are from being free of $a_1, \cdots, a_v$. At the end of this section we will show that under appropriate circumstances relative pushing preserves freedom as measured by a marker cascade. The next section shows that judicious relative pushing increases the level of freedom. See Proposition 9.1.

Definition 8.1. Let V be the underlying space of a finite simplicial complex. Associated to $f : V \to \mathcal{PMLEL}(S)$ a generic PL map, $\mathcal{J} = (\mathcal{M}_1, W_1), \cdots, (\mathcal{M}_m, W_m)$ a marker family hit by f, $\mathcal{M}_1 < \cdots < \mathcal{M}_m$ the ordering induced from this enumeration and $a_1, \cdots, a_v$ a sequence of pairwise disjoint simple closed geodesics we define a *marker cascade* $\mathcal{C}$ which is a $v+1$-tuple $(\mathcal{A}_1, \cdots, \mathcal{A}_v, \mathcal{P})$ where each $\mathcal{A}_i$ is a 3-tuple $(\mathcal{A}_i(i), \mathcal{A}_i(ii), \mathcal{A}_i(iii))$ that is defined below and $\mathcal{P}$ is a finite set of v-tuples defined in Definition 8.5. To start with $\mathcal{A}_i$ is organized as follows.

$\mathcal{A}_i(i)$ is either a marker $\mathcal{M}_{i_j}$ or ∞.

$\mathcal{A}_i(ii) \in \mathbb{Z}_{\ge 0} \cup \infty$ is the geometric intersection number of f with the tag τ_i associated to a_i and $\mathcal{M}_{i_j}$ along the compact set $m_i(\mathcal{C}) \subset V$, unless $\mathcal{A}_i(i) = \infty$ in which case $\mathcal{A}_i(ii) = \infty$.

$\mathcal{A}_i(iii)$ is the set of stryker curves for τ_i along the compact set $m_i(\mathcal{C}) \subset V$ unless $\mathcal{A}_i(ii) = \infty$ in which case $\mathcal{A}_i(iii) = \infty$.

We define the $\mathcal{A}_i$'s and the auxiliary $m_i(\mathcal{C})$'s as follows.

$\mathcal{A}_1(i)$ is defined to be the maximal marker $\mathcal{M}_{1_j}$ such that $\mathcal{M}_i$ is f-free of a_1 along W_i for all $i < 1_j$. If a_1 is free of $\mathcal{J}$, then define $\mathcal{A}_1(i) = \infty$.

If $\mathcal{M}_{1_j}$ exists, then define τ_1 to be the marker tag associated to a_1 and $\mathcal{M}_{1_j}$ and define $m_1(\mathcal{C}) = \{t \in S(f, a_1, \mathcal{M}_{1_j}, W_{1_j}) | g(f, \tau_1, S(f, a_1, \mathcal{M}_{1_j}, W_{1_j})) = |(\phi(f(t)) \setminus a_1) \cap \tau_1|\}$.

$\mathcal{A}_1(ii) = g(f, \tau_1, m_1(\mathcal{C}))$ if $m_1(\mathcal{C}) \ne \emptyset$ or $\mathcal{A}_1(ii) = \infty$ otherwise.

$\mathcal{A}_1(iii)$ is defined to be Stryker_1 the set of $m_1(\mathcal{C})$)-stryker curves for τ_1, unless $\mathcal{A}_1(ii) = \infty$ in which case $\mathcal{A}_1(iii) = \infty$.

Having defined $\mathcal{A}_i, i < u$, then $\mathcal{A}_u$ is defined as follows. (The reader is encouraged to first read Remark 8.2). To start with define $B_u^1, \cdots, B_u^m$ where $B_u^r = \{b_1^r, \cdots, b_u^r\}$, where $b_u^r = a_u$ and for $q < u, b_q^r = a_q$ if $r < q_j$ and $b_q^r = \emptyset$ otherwise.

$\mathcal{A}_u(i)$ is defined to be either the maximal marker $\mathcal{M}_{u_j}$, such that $r < u_j$ implies that $\mathcal{M}_r$ is free of B_u^r along $m_{u-1}(\mathcal{C}) \cap W_r$ or $\mathcal{A}_u(i) = \infty$ if for all $r \le m$, $\mathcal{M}_r$ is free of B_u^r along $m_{u-1}(\mathcal{C}) \cap W_r$.

Remark 8.2. In words $\mathcal{A}_u(i) = \mathcal{M}_{u_j}$ is the maximal marker such that all lower markers are free of $\{a_1, \cdots, a_u\}$, *where applicable.* Where applicable means two things. First, the only relevant points are those of $m_{u-1}(\mathcal{C})$. Second if say $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ are free of a_1 but $\mathcal{M}_4$ is not and along $m_1(\mathcal{C})$, $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ are free of $\{a_1, a_2\}$ and $\mathcal{M}_4, \mathcal{M}_5$ are free of a_2 but $\mathcal{M}_6$ is not free of a_2, then $\mathcal{M}_{2_j} = \mathcal{M}_6$. In particular, a_1 is irrelevant when considering $\mathcal{M}_p$, for $p \geq 4$. In this case $B_2^1 = B_2^2 = B_2^3 = \{a_1, a_2\}$ and for $r > 3$, $B_2^r = \{a_2\}$. Note that if $\mathcal{M}_1$ is free of $\{a_1, a_2\}$ along $m_1(\mathcal{C})$ but $\mathcal{M}_2$ is not, then $\mathcal{M}_{2_j} = \mathcal{M}_2$.

Definition 8.1 continued If $\mathcal{M}_{u_j}$ exists, then define τ_u to be the marker tag arising from a_u and $\mathcal{M}_{u_j}$. Let $S_u = S(f, B_u^{u_j}, \mathcal{M}_{u_j}, W_{u_j}) \cap m_{u-1}(\mathcal{C})$. Define $m_u(\mathcal{C}) = \{t \in S_u | g(f, \tau_u, S_u) = |\phi(f_t) \setminus a_u) \cap \tau_u|\}$.

$\mathcal{A}_u(ii)$ is defined to be either $g(f, \tau_u, m_u(\mathcal{C}))$ or ∞ if $m_u(\mathcal{C}) = \emptyset$.

$\mathcal{A}_u(iii)$ is defined to be the set Stryker$_u$ which is either the set of $m_u(\mathcal{C}))$-stryker curves for τ_u if $m_u(\mathcal{C}) \neq \emptyset$ or ∞ otherwise.

We say that the cascade $\mathcal{C}$ is *finished* if $m_v(\mathcal{C}) = \emptyset$ and *active* otherwise. We say that the cascade is *based* on $\{a_1, \cdots, a_v\}$ and has *length* v. For $r \leq v$, then the length-r cascade based on $\{a_1, \cdots, a_r\}$ is called the length-r *subcascade* and denoted $\mathcal{C}_r$. Note that $\mathcal{C}_r$ and $\mathcal{C}$ have the same values of $\mathcal{A}_1, \cdots, \mathcal{A}_r$.

Notation 8.3. The data corresponding to a cascade depends on f. When the function must to be explicitly stated, we will use notation such as $\mathcal{C}(f), m_i(\mathcal{C}, f), \mathcal{A}_p(f)$ or $\mathcal{A}_r(ii, f)$.

We record for later use the following result.

Lemma 8.4. *Let $\mathcal{J}$ be a marker family hit by the generic PL map $f : V \to \mathcal{PMLEL}(S)$. If $\mathcal{C}$ is an active cascade based on $a_1, \cdots, a_v$, then for every $t \in m_v(\mathcal{C})$, each a_i is a leaf of $\phi(f(t))$.*

Proof. By definition $m_1(\mathcal{C}) \subset S(f, a_1, \mathcal{M}_{1_j}, W_{1_j})$, hence a_1 is a leaf of $\phi(f(t))$ at all $t \in m_1(\mathcal{C})$.

Now assume the lemma is true for all subcascades of length $< u$. Let $t \in m_u(\mathcal{C})$. Let $A = B_u^{u_j}$. By definition, $\mathcal{M}_{u_j}$ is not f-free of A along $m_{u-1}(\mathcal{C})$, but $\mathcal{M}_{u_j}$ is f-free of $A \setminus a_u$ along $m_{u-1}(\mathcal{C})$. Since $m_u(\mathcal{C}) \subset S(f, A, W_{u_j}) \cap m_{u-1}(\mathcal{C})$ it follows that a_u is a leaf of $\phi(f(t))$. $\square$

Definition 8.5. Let $\mathcal{C}$ be an active cascade. To each $t \in m_v(\mathcal{C})$ corresponds a v-tuple $(p_1, \cdots, p_v)$ where p_j is the (possibly empty) stryker multi-geodesic for τ_j at t. Such a $(p_1, \cdots, p_v)$ is called a *packet.* There are only finitely many packets, by the finiteness of stryker curves. Thus $m_v(\mathcal{C})$ canonically decomposes into a disjoint union of closed sets $S_1, \cdots, S_r$ such that each point in a given S_j has the same packet. Let $\mathcal{P} = \{P_1, \cdots, P_r\}$ denote the set of packets, the last entry in the definition of $\mathcal{C}$. We will use the notation $\mathcal{P}(f)$, when needed to clarify the function on which this information is based.

Definition 8.6. In what follows all cascades use the same set of markers and simple closed geodesics, however the function $f : V \to \mathcal{PMLEL}(S)$ will vary. We put an equivalence relation on this set of cascades and then partially order the classes. We say that $\mathcal{C}(g)$ is equivalent to $\mathcal{C}(f)$ if $\mathcal{P}(f) = \mathcal{P}(g)$ and for all r, $\mathcal{A}_r(f) = \mathcal{A}_r(g)$. We lexicographically partial order the equivalence classes by comparing the v-tuples $(\mathcal{A}_1(\mathcal{C}(f)), \cdots, \mathcal{A}_v(\mathcal{C}(f)), \mathcal{P}(f))$ using the rule that $\mathcal{A}_r(i,f) \le \mathcal{A}_r(i,g)$ if $\mathcal{M}_{r_j}(f) \le \mathcal{M}_{r_j}(g)$, with ∞ being considered the maximal value and $\mathcal{A}_r(ii,f) \le \mathcal{A}_r(ii,g)$ if their values satisfy that inequality and $\mathcal{A}_r(iii,f) \le \mathcal{A}_r(iii,g)$ if $\text{Stryker}_r(g) \subset \text{Stryker}_r(f)$. Finally, $\mathcal{P}(f) \le \mathcal{P}(g)$ if $\mathcal{P}(g) \subset \mathcal{P}(f)$.

Remark 8.7. More or less, $\mathcal{C}(f) < \mathcal{C}(g)$ means that the markers's are freer with respect to the function g than with respect to f. In particular, this inequality holds if $\mathcal{M}_1$ and $\mathcal{M}_2$ are g-free of a_1, but only $\mathcal{M}_1$ and f-free of a_1. If $\mathcal{M}_1$ is both g-free and f-free of a_1 but $\mathcal{M}_2$ is neither f-free nor g-free of a_1, then $\mathcal{A}_1(ii)$ is a measure of how close $\mathcal{M}_2$ is from being free of a_1. The bigger the number, the closer to freedom as motivated by Lemma 7.2. If this number is the same with respect to both f and g, then $\mathcal{A}(iii)$ measures how much work is needed to raise the number. More stryker multi-geodesics with respect to f, than g, means more needs to be done to f, so again $\mathcal{C}(f) < \mathcal{C}(g)$. If $\mathcal{A}_1(f) = \mathcal{A}_1(g)$, then $\mathcal{A}_2$ is used to determine the ordering. Finally, if all the $\mathcal{A}_i$'s are equal, then f having more packets means that *more sets* of V need to be cleaned up to finish the cascade.

Proposition 8.8. *Let V be the underlying space of a finite simplical complex. Let $f : V \to \mathcal{PMLEL}(S)$ a generic PL map, $\dim(V) \le n$ and $\mathcal{J} = (\mathcal{M}_1, W_1), \cdots, (\mathcal{M}_m, W_m)$ a marker family hit by f. Let $\mathcal{C}$ be an active cascade based on $\{a_1, \cdots, a_v\}$, C a simple closed geodesic disjoint from the a_i's and the τ_i's and $K \subset m_v(\mathcal{C}) \cap f^{-1}(B_C)$ compact. For $1 \le i \le m$, let $B_C^i = \{a_u | i < u_j\} \cup \{C\}$. Assume that for $1 \le i \le m$, $\mathcal{M}_i$ is f-free of B_C^i along $K \cap W_i$. If δ is sufficiently small and f_1 is obtained from f by a (C, δ, K) push off homotopy, then $[\mathcal{C}(f)] \le [\mathcal{C}(f_1)]$ and the homotopy from f to f_1 is $\mathcal{J}$-marker preserving.*

Proof. Since $\mathcal{J}$ is f-free of C along K, it follows by Lemma 6.15 that for δ sufficiently small, any (C, δ, K) homotopy is $\mathcal{J}$ marker preserving. It remains to show that if δ is sufficiently small and f_1 is the resulting map, then $[\mathcal{C}(f)] \le [\mathcal{C}(f_1)]$.

From $\mathcal{C}(f)$ we conclude that for all $u \in \{1, \cdots, v\}$ and $q < u_j$, $\mathcal{M}_q$ is f-free of B_u^q along $m_v(\mathcal{C}) \cap W_q$. By hypothesis each $\mathcal{M}_q$ is also f-free of B_C^q along $W_q \cap K$. Note that $B_u^q \subset B_C^q$ when $q < u_j$. By Lemma 6.15, if δ is sufficiently small, then there exists a neighborhood V of K such that any (C, δ, K) homotopy is supported in V and each $\mathcal{M}_q$ is F-free of B_C^q along $W_q \cap \bar{V}$. It follows that with respect to the lexicographical ordering $(\mathcal{A}_1(i,f), \cdots, \mathcal{A}_v(i,f)) \le (\mathcal{A}_1(i,f_1), \cdots, \mathcal{A}_v(i,f_1))$.

A similar argument using Lemma 6.19 shows that if δ is sufficiently small and $\mathcal{A}_i(i,f) = \mathcal{A}_i(i,f_1)$ for $1 \le i \le u$, then $\mathcal{A}_i(ii,f) \le \mathcal{A}_i(ii,f_1)$ for $1 \le i \le u$.

By Lemma 7.4 there exists $\eta > 0$ such that if $d_V(t_1, t_2) < 2\eta$ and $t_1, t_2 \in m_u(\mathcal{C})$ for some u, then $f(t_1), f(t_2)$ have the same stryker curve to τ_u. Let δ be sufficiently small to satisfy Lemma 7.5 with this η in addition to the previously required conditions. That lemma implies that if $\mathcal{A}_i(i,f) = \mathcal{A}_i(i,f_1)$ and $\mathcal{A}_i(ii,f) = \mathcal{A}_i(ii,f_1)$ for all $i \le u$, then $\text{Stryker}_i(f_1) \subset \text{Stryker}_i(f)$ for all $i \le u$.

Finally, Lemma 7.5 with this choice of η also implies that if $(\mathcal{A}_1(f), \cdots, \mathcal{A}_v(f)) = (\mathcal{A}_1(f_1), \cdots, \mathcal{A}_v(f_1))$, then $\mathcal{P}(f_1) \subset \mathcal{P}(f)$. It follows that $[\mathcal{C}(f)] \le [\mathcal{C}(f_1)]$. ☐

Remark 8.9. Note that $[\mathcal{C}(f)] \le [\mathcal{C}(f_1)]$ holds with respect to the lexicographical ordering, but may not hold entry-wise, since there may be no direct comparison between later entries once earlier ones differ. For example, say $2 = \mathcal{A}_1(i,f) < \mathcal{A}_1(i,f_1) = 3$, then showing $\mathcal{A}_2(i,f) = 3$ involves verifying that $\mathcal{M}_2$ is f-free of $\{a_2\}$ while showing $\mathcal{A}_2(i,f_1) = 3$ involves verifying that $\mathcal{M}_2$ is f_1-free of $\{a_1, a_2\}$.

9. Finishing Cascades

The main result of this section is the following.

Proposition 9.1. *Let V be the underlying space of a finite simplicial complexh : $V \to \mathcal{PMLEL}(S)$ be a generic PL map such that $k = \dim(V) \leq n$ where dim$(\mathcal{PML}(S)) = 2n+1$. Let $\mathcal{J}$ be a marker family hit by h and $\mathcal{C}$ be an active cascade. Then there exists a marker preserving homotopy of h to h' such that $\mathcal{C}$ is finished with respect to h' and $[\mathcal{C}(h)] < [\mathcal{C}(h')]$. The homotopy is a concatenation of relative push offs. If $L \subset \mathcal{PML}(S)$ is a finite subcomplex of $\mathcal{C}(S)$, then the homotopy can be chosen to be disjoint from L.*

Lemma 9.2. *Let $f_i : V \to \mathcal{PMLEL}(S)$, $i \in \mathbb{N}$ be generic PL maps, $\mathcal{J}$ be a marker family and let $\{\mathcal{C}(f_i)\}$ be active cascades based on the same set of simple closed geodesics. Any sequence $[\mathcal{C}(f_1)] \leq [\mathcal{C}(f_2)] \leq \cdots$ has only finitely many terms that are strict inequalities.*

Proof. There are only finitely many possible values for $\mathcal{A}_u(i, f_j)$.

It follows from Lemma 7.2 that each $\mathcal{A}_u(ii, f_j)$ is uniformly bounded above. Indeed, there are only finitely many a_i's and only finitely many markers in $\mathcal{J}$, thus only finitely many marker tags arising from them. If τ_u is such a tag, then for $i \in \mathbb{N}$, $g(f_i, \tau, m_u(\mathcal{C}(f_i))) \leq |(\cup_{j=1}^{v} a_j) \cap \tau_u| + 3(3g-3+2p)$.

The finiteness of stryker curves (Lemma 4.5) shows that the number of possibilities for both $\mathcal{P}(f_j)$ and each $\mathcal{A}_u(iii, f_j)$ is bounded. ∎

Proof of Proposition 9.1. It suffices to show that given any active cascade $\mathcal{C}(h_1)$, there exists a $\mathcal{J}$ marker preserving homotopy from h_1 to h_2, that is a concatenation of relative push offs, such that $[\mathcal{C}(h_1)] < [\mathcal{C}(h_2)]$. For if $\mathcal{C}(h_2)$ is not finished, then we can similarly produce an h_3 with $[\mathcal{C}(h_2)] < [\mathcal{C}(h_3)]$. Eventually we obtain a finished $\mathcal{C}(h_q)$ else we contradict Lemma 9.2.

We retain the convention that $\dim(\mathcal{PML}(S)) = 2n+1$. We will assume that $k = n$ leaving the easier $k < n$ case to the reader. The proof is by downward induction on $L = \text{length}(\mathcal{C})$. Suppose that $\mathcal{C}(h)$ is an active cascade based on $\{a_1, \cdots, a_L\}$. Since the a_i's are pairwise disjoint, it follows that $L \leq n+1$. Let τ_i denote the marker tag associated to a_i and $\mathcal{M}_{i_j}(h)$. Let $R' = N(a_1 \cup \cdots \cup a_L \cup \tau_1 \cdots \cup \tau_L)$. Let R be R' together with all its complementary discs, annuli and pants. Let $T = S \setminus (\text{int}(R))$. Note that $T = \emptyset$ if $L \geq n$.

Let $\mathcal{A}(h(t))$ denote the arational sublamination of $\phi(h(t))$, i.e. the sublamination obtained by deleting all the compact leaves. Since h is generic and $k = n$, it follows that $\mathcal{A}(h(t)) \neq \emptyset$ for all $t \in B^n$. The key observation is that if $\mathcal{A}(h(t)) \cap R \neq \emptyset$, then $t \notin m_L(\mathcal{C}(h))$. To start with, Lemma 8.4 implies that $a_1, \cdots, a_L$ are leaves of $\phi(h(t))$. This implies that $\mathcal{A}(h(t)) \cap \tau_i \neq \emptyset$, where τ_i is the marker tag between a_i and $\mathcal{M}_{i_j}$. By Lemma 7.2 $\phi(h(t))$ hits $\mathcal{M}_{i_j}$ and hence $t \notin m_i(\mathcal{C}(h))$contradicting the fact that $m_L(\mathcal{C}(h)) \subset m_i(\mathcal{C}(h))$. It follows that $\mathcal{C}(h)$ is finished if $L \geq n$.

Note that if $k < n$, then this argument shows that $\mathcal{C}$ is finished if $L \geq k$. In particular, if $k = 1$, (the path connectivity case) all cascades are finished.

Now assume that the Proposition is true for all cascades of length greater than L, where $L < n$. Let $\mathcal{C}$ be an active cascade of length L.

Let $P = (p_1, \cdots, p_L) \in \mathcal{P}(h)$ and S_P the closed subset of $m_L(\mathcal{C})$ consisting of points whose packet is P. Let σ be the possibly empty multi-geodesic $p_1 \cup \cdots \cup p_L$. By definition, if $t \in S_P$, then each p_i is a leaf of $\phi(h(t))$ and by Lemma 8.4 each of

$a_1, \cdots, a_L$ is a leaf of $\phi(h(t))$. Let $Y' = N(a_1 \cup \cdots \cup a_L \cup \tau_1 \cdots \cup \tau_L \cup \sigma)$ and Y be Y' together with all complementary discs, annuli and pants. Let $Z = S \setminus \mathrm{int}(Y)$. Note that $Z \neq \emptyset$, else for all $t \in S_P$, $|\phi(h(t)) \cap \tau_{i(t)}| = \infty$ for some $i(t)$ which is a contradiction as before. Let C be a simple closed geodesic isotopic to some component of ∂Z.

Observe that C is neither an a_i nor is $C \subset \sigma$. Indeed, since each a_i is crossed transversely by a tag at an interior point it follows that no a_i is isotopic to a component of $\partial Y'$ and hence ∂Y. Similarly, each component of Stryker_i is transverse to τ_i at an interior point, hence no component of σ is isotopic to a component of ∂Y. Next observe that $S_P \subset f^{-1}(B_C)$. Indeed, $t \in S_P$ implies that for all $i, \phi(h(t)) \cap \tau_i \subset a_i \cup \sigma$, hence C is either a leaf of $\phi(h(t))$ or $C \cap \phi(h(t)) = \emptyset$.

Extend the cascade $\mathcal{C}(h)$ to the length $L+1$ cascade $\mathcal{C}'(h)$ by letting C be our added a_{L+1}. If $\mathcal{C}'(h)$ is active, then by induction, there is a marker preserving homotopy of h to h_1 so that $\mathcal{C}'(h_1)$ is finished and $[\mathcal{C}'(h)] < [\mathcal{C}'(h_1)]$. If $\mathcal{C}'(h)$ is finished, then let's unify notation by denoting h by h_1. In both cases, by restricting to the length L subcascade either $[\mathcal{C}(h)] < [\mathcal{C}(h_1)]$ or $[\mathcal{C}(h)] = [\mathcal{C}(h_1)]$ and each $\mathcal{M}_i$ is h_1-free of B^i_C for all $t \in m_L(\mathcal{C}(h_1)) \cap W_i$, where B^i_C is as in the statement of Lemma 8.8. Since freedom is an open condition and $S_P(h_1)$ is compact as are all the W_i's, there exists $U \subset V$ open such that $m_L(\mathcal{C}(h_1)) \subset U$ and each $\mathcal{M}_i$ is h_1-free of B^i_C for all $t \in \bar{U} \cap W_i$.

If $[\mathcal{C}(h)] = [\mathcal{C}(h_1)]$, then invoke Lemma 8.8 by taking $f = h_1$, keeping the original $\mathcal{J}$ and $\mathcal{C}$, using the above constructed C and letting $K = S_P(h_1)$. Choose δ sufficiently small to satisfy the conclusion of Lemma 8.8 and so that any (C, δ, K) push off homotopy is supported in U.

To complete the proof it suffices to show that if δ is sufficiently small and h_2 is the resulting map, then $[\mathcal{C}(h_1)] < [\mathcal{C}(h_2)]$. Since Lemma 8.8 implies that $[\mathcal{C}(h_1)] \leq [\mathcal{C}(h_2)]$ it suffices to show that $P \notin \mathcal{P}(h_2)$ if for all i, $\mathcal{A}_i(h_1) = \mathcal{A}_i(h_2)$. If $P \in \mathcal{P}(h_2)$, then let $t \in S_P(h_2)$. As above $h_2(t) \in B_C$. Since h_2 is the result of a relative push off homotopy it follows from the last sentence of Definition 6.12 iii) that either $\phi(h_2(t)) = \phi(h_1(t))$ or $\phi(h_2(t)) \neq \phi(h_1(t))$ but $C \cup \phi(h_2(t)) = \phi(h_1(t))$. In the former case, $t \in S_P(h_1)$ and hence both $h_1(t), h_2(t) \in B_C$, contradicting the fact that h_2 is the result of a (C, δ, K) push off homotopy. In the latter case $t \in U \setminus S_P(h_1)$, hence there exists an $\mathcal{M}_i$ that is h_1-free of $B^i_C \setminus C$ at $t \in W_i$, but $\mathcal{M}_i$ is not h_2-free of $B^i_C \setminus C$ at t. This implies that $\mathcal{M}_i$ is not h_1-free of B^i_C at t which contradicts the fact that $t \in U$.

Since $k \leq n$ Lemma 2.4 it follows that $h(V) \cap K = \emptyset$. By Lemma 6.14, by using sufficiently small δ's all (C, δ, K) pushoff homotopies in the above proof could have been done to avoid L. $\square$

Corollary 9.3. *Let V be the underlying space of a finite simplicial complex. Let S be a finite type surface and $f : V \to \mathcal{PMLEL}(S)$ be a generic PL map and $\dim(V) \leq n$, where $\dim(\mathcal{PML}(S)) = 2n+1$. Let $\mathcal{J}$ be a marker family hit by f and C a simple closed geodesic. Then f is homotopic to f_1 via a marker preserving homotopy such that $g(f, C) > 0$. If $L \subset \mathcal{PML}(S)$ is a finite subcomplex of $\mathcal{C}(S)$, then the homotopy can be chosen to be disjoint from L.*

Proof. By Lemma 6.6, if $\mathcal{J}$ is free of C and δ is sufficiently small, then any (C, δ) push off homotopy is $\mathcal{J}$-marker preserving. If f_1 is the resulting map, then $g(f_1, C) > 0$.

If $\mathcal{J}$ is not free of C, then let $\mathcal{C}$ be the active length-1 cascade based on C. By Proposition 9.1 f is homotopic to f' by a marker preserving homotopy such that $\mathcal{C}(f')$ is finished and the homotopy is a concatenation of relative push off homotopies. Now argue as in the first paragraph. $\square$

10. Stryker Cascades

The main result of this section is

Proposition 10.1. *Let V be the underlying space of a finite simplicial complex. Let $f : V \to \mathcal{PML}\mathcal{EL}(S)$ be a generic PL map such that* $\dim(V) \leq n$ *where* $\dim(\mathcal{PML}(S)) = 2n+1$. *If a_1 is a simple closed geodesic such that $\infty > g(f, a_1) > 0$ and $\mathcal{J}$ a marker family hit by f, then there exists then there exists a marker preserving homotopy of f to f_1 such that $g(f_1, a_1) > g(f, a_1)$. The homotopy is a concatenation of relative push offs. If $L \subset \mathcal{PML}(S)$ is a finite subcomplex of $\mathcal{C}(S)$, then the homotopy can be chosen to be disjoint from L.*

Remarks 10.2. The proof is very similar to that of Corollary 9.3, except that *stryker cascades* are used in place of marker cascades. A stryker cascade is essentially a marker cascade except that the first term is a curve a_1 with $g(f, a_1) > 0$.

Closely following the previous two sections, we give the definition of stryker cascade and prove various results about them.

Definition 10.3. Associated to $f : V \to \mathcal{PML}\mathcal{EL}(S)$ a generic PL map, $\mathcal{J} = (\mathcal{M}_1, W_1), \cdots, (\mathcal{M}_m, W_m)$ a marker family hit by f, $\mathcal{M}_1 < \cdots < \mathcal{M}_m$ the ordering induced from this enumeration and $a_1, \cdots, a_v$ a sequence of pairwise disjoint simple closed geodesics with $g(f, a_1) > 0$ we define a *stryker cascade* $\mathcal{C}$ which is a $v+1$-tuple $(\mathcal{A}_1, \cdots, \mathcal{A}_v, \mathcal{P})$ where $\mathcal{A}_1$ is a 2-tuple $(\mathcal{A}_1(i), \mathcal{A}_1(ii))$ and for $i \geq 2$, each $\mathcal{A}_i$ is a 3-tuple $(\mathcal{A}_i(i), \mathcal{A}_i(ii), \mathcal{A}_i(iii))$ and $\mathcal{P}$ is a finite set of v-tuples defined in Definition 10.6.

Define $\mathcal{A}_1 = (g(f, a_1), \mathrm{Stryker}_1)$ where $\mathrm{Stryker}_1$ is the set of stryker curves to f and a_1.

Define $m_1(\mathcal{C}) = \{t \in V | g(f, a_1) = |\phi(f(t)) \cap a_1|\}$.

Having defined $\mathcal{A}_i, 1 \leq i < u$, then $\mathcal{A}_u$ is defined as follows. To start with define $B_u^1, \cdots, B_u^m$ where $B_u^r = \{b_2^r, \cdots, b_u^r\}$, where $b_u^r = a_u$ and for $2 \geq q < u, b_q^r = a_q$ if $r < q_j$ and $b_q^r = \emptyset$ otherwise.

$\mathcal{A}_u(i)$ is defined to be either the maximal marker $\mathcal{M}_{u_j}$, such that $r < u_j$ implies that $\mathcal{M}_r$ is free of B_u^r along $m_{u-1}(\mathcal{C}) \cap W_r$ or $\mathcal{A}_u(i) = \infty$ if for all $r \leq m$, $\mathcal{M}_r$ is free of B_u^r along $m_{u-1}(\mathcal{C}) \cap W_r$.

If $\mathcal{M}_{u_j}$ exists, then define τ_u to be the marker tag arising from a_u and $\mathcal{M}_{u_j}$. Let $S_u = S(f, B_u^{u_j}, \mathcal{M}_{u_j}, W_{u_j}) \cap m_{u-1}(\mathcal{C})$. Define $m_u(\mathcal{C}) = \{t \in S_u | g(f, \tau_u, S_u) = |\phi(f_t) \setminus a_u) \cap \tau_u|\}$.

$\mathcal{A}_u(ii)$ is defined to be either $g(f, \tau_u, m_u(\mathcal{C}))$ or ∞ if $m_u(\mathcal{C}) = \emptyset$.

$\mathcal{A}_u(iii)$ is defined to be the set $\mathrm{Stryker}_u$ which is either the set of $m_u(\mathcal{C}))$-stryker curves for τ_u if $m_u(\mathcal{C}) \neq \emptyset$ or ∞ otherwise.

We say that the cascade $\mathcal{C}$ is *finished* if $m_v(\mathcal{C}) = \emptyset$ and *active* otherwise. We say that the cascade is *based* on $\{a_1, \cdots, a_v\}$ and has *length* v. For $r \leq v$, then length-q cascade based on $\{a_1, \cdots, a_r\}$ is called the length-r *subcascade* and denoted $\mathcal{C}_r$. Note that $\mathcal{C}_r$ and $\mathcal{C}$ have the same values of $\mathcal{A}_1, \cdots, \mathcal{A}_r$.

Notation 10.4. The data corresponding to a cascade depends on f. When the function must to be explicitly stated, we will use notation such as $\mathcal{C}(f), m_i(\mathcal{C}, f), \mathcal{A}_p(f)$ or $\mathcal{A}_r(ii, f)$.

We record for later use the following result whose proof is essentially identical to that of Lemma 8.4.

Lemma 10.5. *Let $\mathcal{J}$ be a marker family hit by the generic PL map $f : V \to \mathcal{PMLEL}(S)$. If $\mathcal{C}$ is an active stryker cascade based on $a_1, \cdots, a_v$, then for every $t \in m_v(\mathcal{C})$ and $i \geq 2$, each a_i is a leaf of $\phi(f(t))$.* □

Definition 10.6. Let $\mathcal{C}$ be an active stryker cascade. To each $t \in m_v(\mathcal{C})$ corresponds an v-tuple $(p_1, \cdots, p_v)$ where p_1 is the stryker multi-geodesic for a_1 at t and for $i \geq 2$, p_i is the (possibly empty) stryker multi-geodesic for τ_i at t. Such a $(p_1, \cdots, p_v)$ is called a *packet*. There are only finitely many packets, by the finiteness of stryker curves. Thus $m_v(\mathcal{C})$ canonically decomposes into a disjoint union of closed sets $S_1, \cdots, S_r$ such that each point in a given S_j has the same packet. Let $\mathcal{P} = \{P_1, \cdots, P_r\}$ denote the set of packets, the last entry in the definition of $\mathcal{C}$.

Definition 10.7. Use the direct analogy of Definition 8.6 to put an equivalence relation on the set of stryker cascades based on the same ordered set of simple closed curves and to partially order the classes.

Proposition 10.8. *Let $f : V \to \mathcal{PMLEL}(S)$ a generic PL map, $\dim(V) \leq n$ and $\mathcal{J} = (\mathcal{M}_1, W_1), \cdots, (\mathcal{M}_m, W_m)$ a marker family $\mathcal{J}$ hit by f. Let $\mathcal{C}$ be an active stryker cascade based on $\{a_1, \cdots, a_q\}, C$ a simple closed geodesic disjoint from the a_i's and τ_j's and let $K \subset m_v \cap f^{-1}(B_C)$. For $1 \leq i \leq m$, let $B^i_C = \{a_p, p \geq 2 | i < p_j\} \cup \{C\}$. Assume that for $1 \leq i \leq m$, $\mathcal{M}_i$ is f-free of B^i_C along $K \cap W_i$. If f_1 is obtained from f by a (C, δ, K) push off homotopy and δ is sufficiently small, then $[\mathcal{C}(f)] \leq [\mathcal{C}(f_1)]$ and the homotopy from f to f_1 is $\mathcal{J}$-marker preserving.*

Proof. The fact $\mathcal{A}_1(f) \leq \mathcal{A}_1(f_1)$ follows from Lemma 6.19. The rest of the argument follows as in the proof of Proposition 8.8. □

We have the following analogue of Lemma 9.2.

Lemma 10.9. *Let $f_i : V \to \mathcal{PMLEL}(S)$, $i \in \mathbb{N}$ be generic PL maps, $\mathcal{J}$ a marker family and let $\{\mathcal{C}(f_i)\}$ be active stryker cascades based on the same set of simple closed geodesics. Any sequence $[\mathcal{C}(f_1)] \leq [\mathcal{C}(f_2)] \leq \cdots$ with $g(f_1, a_1) = g(f_2, a_1) = \cdots$ has only finitely many terms that are strict inequalities.* □

Proposition 10.10. *Let $h : V \to \mathcal{PMLEL}(S)$ be a generic PL map such that $k = \dim(V) \leq n$ where $\dim(\mathcal{PML}(S)) = 2n + 1$. Let $\mathcal{J}$ be a marker family hit by h and let $\mathcal{C}$ be an active stryker cascade with $\infty > g(h, a_1) > 0$. Then there exists a marker preserving homotopy of h to h' such that $[\mathcal{C}(h)] < [\mathcal{C}(h')]$ and either $g(h', a_1) > g(h, a_1)$ or $\mathcal{C}(h')$ is finished. The homotopy is a concatenation of relative push offs. If $L \subset \mathcal{PML}(S)$ is a finite subcomplex of $\mathcal{C}(S)$, then the homotopy can be chosen to be disjoint from L.*

Proof. As in the proof of Proposition 9.1 it suffices to show that given any active stryker cascade $\mathcal{C}(h_1)$ there exists a marker preserving homotopy of h_1 to h_2, that is a concatenation of relative push offs, such that $[\mathcal{C}(h_1)] < [\mathcal{C}(h_2)]$. Here we invoke Lemma 10.9 instead of Lemma 9.2.

Again we discuss the k=n case, the easier $k < n$ cases being left to the reader. The proof by downward induction on the length of the cascade follows essentially exactly that of Proposition 9.1 until the step of proving it for length-1 cascades. So now assume that the Proposition has been proved for cascades of length ≥ 2.

Let $\mathcal{C}(h)$ be a length-1 cascade based on a_1. Here $\mathcal{P}$, the set of packets, consists of the set of stryker multi-curves to h and a_1. Let P be one such multi-curve. Let $Y' = N(a_1 \cup P)$ and Y be the union of Y' and all components of $S \setminus Y'$ that are discs, annuli and pants. Again, genericity and the condition $k = n$ implies that $Y \neq S$. Let C be a simple closed geodesic isotopic to a component of ∂Y. The condition $g(h, a_1) > 0$ implies that C is not isotopic to a_1. Since each component of P non trivially intersects a_1, C is not isotopic to any component of P. Let $\mathcal{C}'$ denote the length-2 stryker cascade based on $\{a_2, C\}$. By induction there exists a marker preserving homotopy of h to h_1 that is a concatenation of relative push offs such that $[\mathcal{C}'(h)] < [\mathcal{C}'(h_1)]$ and either $g(h_1, a_1) > g(h, a_1)$ or $\mathcal{C}'(h_1)$ is finished. In the latter case we have either $[\mathcal{C}(h)] < [\mathcal{C}(h_1)]$ or $[\mathcal{C}(h)] = [\mathcal{C}(h_1)]$ and for each $1 \leq i \leq m$, $\mathcal{M}_i$ is free of C along $m_1(\mathcal{C}, h_1)$. Now argue as in the proof of Proposition 9.1 that if h_2 is the result of a (C, δ, K) homotopy, where δ is sufficiently small and $K = S_P$, then either $g(h_2, a_1) > g(h_1, a_1)$ or $g(h_2, a_1) = g(h_1, a_1)$ and $[\mathcal{C}(h_1)] < [\mathcal{C}(h_2)]$. $\square$

Proof of Proposition 10.1. Let $\mathcal{C}(f)$ be the length-1 stryker cascade based on a_1. By Proposition 10.10 there exists a marker preserving homotopy from f to f_1 that is the concatenation of relative push off homotopies such that $\mathcal{C}(f) < \mathcal{C}(f_1)$. Therefore, either $g(f_1, a_1) > g(f, a)$ or equality holds and $|\mathcal{P}(\mathcal{C}(f_1))| < |\mathcal{P}(\mathcal{C}(f))|$. After finitely many such homotopies we obtain f' such that $g(f', a_1) > g(f_1, a)$. $\square$

11. $\mathcal{EL}(S)$ is (n-1)-connected

Theorem 11.1. *Let V be the underlying space of a finite simplicial complex and W that of a subcomplex. Let S be a finite type hyperbolic surface with* $\dim(\mathcal{PML}(S)) = 2n+1$. *If $f : V \to \mathcal{PMLEL}(S)$ is a generic PL map, $f^{-1}(\mathcal{EL}(S)) = W$, $\dim(V) \leq n$ and $\mathcal{J}$ is a marker family hit by f, then there exists a map $g : V \to \mathcal{EL}(S)$ such that $g|W = f$ and g hits $\mathcal{J}$. The map g is the concatenation of (possibly infinitely many) relative push offs. If $L \subset \mathcal{PML}(S)$ is a finite subcomplex of $\mathcal{C}(S)$, then the homotopy can be chosen to be disjoint from L.*

Proof. Let $C_1, C_2, \cdots$ be an enumeration of the simple closed geodesics on S. It suffices to find a sequence, $f_0, f_1, f_2, \cdots$ of extensions of $f|W$ and sequences $\{\mathcal{E}_i\}$, $\{\mathcal{S}_i\}$ of marker covers, such that $\mathcal{E}_i$ is a marker covering of V by $1/i$ markers, $\mathcal{S}_i$ is a marker covering of V by C_i markers and for $i \leq j$, f_j hits each $\mathcal{E}_i$ and $\mathcal{S}_i$ family of markers. Furthermore, f_{i+1} is obtained from f_i by a finite sequence of relative push offs. Then Proposition 5.2 produces g as the limit of the f_j's. Since f_{i+1} is obtained from f_i by concatenating finitely many push offs, concatenating all these push off homotopies produces a map $F : V \times [0, \infty] \to \mathcal{PMLEL}(S)$ with $F|V \times \infty = g$. The proof of Proposition 5.2 shows that F is continuous. It follows from Lemma 6.14 that the homotopy can be chosen disjoint from L.

Suppose that $f_0, f_1, \cdots, f_{j-1}, \mathcal{E}_1, \cdots, \mathcal{E}_{j-1}$, and $\mathcal{S}_1, \cdots, \mathcal{S}_{j-1}$ have been constructed so that f_q hits each $\mathcal{E}_p$ and $\mathcal{S}_p$ family of markers whenever $p \leq q \leq j-1$. We will extend the sequence by constructing f_j, $\mathcal{E}_j$ and $\mathcal{S}_j$ to satisfy the corresponding properties. The theorem then follows by induction. In what follows, $\mathcal{J}_i$ denotes the marker family that is the union of all the markers in the $\mathcal{E}_r$ and $\mathcal{S}_s$ marker families, where $r, s \leq i$.

Step 1. Construction of f_j and $\mathcal{S}_j$:

Proof. Let $N(C_j)$ be as in Lemma 3.11 ii). Given f_{j-1}, obtain f_j^1 by applying Corollary 9.3 to f_j and the marker cover $\mathcal{J}_{j-1}$. Now repeatedly apply Proposition 10.1 to obtain $f_j^2, f_j^3, \cdots, f_j^{N(C_j)}$ with the property that f_j^q is a generic PL map obtained from f_{j-1}^{q-1} via a $\mathcal{J}_{j-1}$ marker preserving homotopy such that $g(f_j^q, C) \geq q$. Let $f_j = f_j^{N(C_j)}$ and apply Lemma 3.11 ii) to obtain $\mathcal{S}_j$. $\square$

Step 2. Construction of $\mathcal{E}_j$:

Proof. Apply Lemma 3.11 i) to the generic PL map f_j to obtain $\mathcal{E}_j$ a marker cover by $1/j$ markers. $\square$

Theorem 11.2. *If S is a finite type hyperbolic surface such that* $\dim(\mathcal{PML}(S)) = 2n+1$*, then $\mathcal{EL}(S)$ is $(n-1)$-connected.*

Proof. Let $k \leq n$ and $g : S^{k-1} \to \mathcal{EL}(S)$ be continuous and $f : B^k \to \mathcal{PMLEL}(S)$ be a generic PL extension of g, provided by Proposition 2.7. Now apply Theorem 11.1 to extend g to a map of B^k into $\mathcal{EL}(S)$. $\square$

12. $\mathcal{EL}(S)$ is $(n-1)$-locally connected

Theorem 12.1. *If S is a finite type hyperbolic surface, then $\mathcal{EL}(S)$ is $(n-1)$-locally connected, where* $\dim(\mathcal{PML}(S)) = 2n+1$.

Proof. We need to show that if $\mathcal{L} \in \mathcal{EL}(S)$, and $U' \subset \mathcal{EL}(S)$ is an open set containing $\mathcal{L}$, then there exists an open set $U \subset U'$ containing $\mathcal{L}$ so that if $k \leq n$, $g : S^{k-1} \to \mathcal{EL}(S)$ and $g(S^{k-1}) \subset U$, then g extends to a map $F : B^k \to \mathcal{EL}(S)$ such that $F(B^k) \subset U'$.

Choose a pair of pants decomposition and parametrizations of the the cuffs and pants so that some complete maximal train track τ fully carries $\mathcal{L}$. Thus $V(\tau)$ is a convex subset of $\mathcal{ML}(S)$ that contains $\hat{\phi}^{-1}(\mathcal{L})$ in its interior.

Let $\mathcal{M}_1, \mathcal{M}_2, \cdots$ be a sequence of markers such that each $\mathcal{M}_i$ is a $1/i$-marker that is that is hit by $\mathcal{L}$. Let $U_i = \{x \in \mathcal{EL}(S) | \phi(x) \text{ hits } \mathcal{M}_i\}$. Then each U_i is an open set containing $\mathcal{L}$ and by Lemma 4.5 there exists $N \in \mathbb{N}$ such that if $i \geq N$, then $\hat{\phi}^{-1}(U_i) \subset \text{int}(V(\tau)) \cap \hat{\phi}^{-1}(U')$. Reduce the ends of each post of $\mathcal{M}_N$ slightly to obtain $\mathcal{M}_N^*$ so that if $U_N^* = \{x \in \mathcal{EL}(S) | \phi(x) \text{ hits } \mathcal{M}_N^*\}$ and $\bar{U}_N^* = \{x \in \mathcal{EL}(S) | \phi(x)$ hits $\bar{\mathcal{M}}_N^*\}$, then $\mathcal{L} \in U_N^* \subset \bar{U}_N^* \subset U_N$. Let $\hat{W} \subset \hat{\phi}^{-1}(U_N^*)$ be an open convex subset of $\mathcal{ML}(S)$ that contains $\hat{\phi}^{-1}(\mathcal{L})$ and is saturated by rays through the origin. By reducing $\hat{W}$ we can assume that $\hat{W} = \text{int}(V(\tau))$ for some train track τ carrying $\mathcal{L}$. Next choose j such that $\hat{\phi}^{-1}(U_j) \subset \hat{W}$. Let $U = U_j$ and $\mathcal{M} = \mathcal{M}_j$.

Let $k \leq n$ and $g : S^{k-1} \to \mathcal{EL}(S)$ with $g(S^{k-1}) \subset U$. Since $\hat{W}$ is convex and contains $\hat{\phi}^{-1}(U)$, we can apply Proposition 2.7 and Remark 2.19 to find a generic PL map $f_0 : B^k \to \mathcal{PMLEL}(S)$ extending g such that $f_0(\text{int}(B^k)) \subset W$. Thus for all $t \in B^k, f_0(t)$ hits $\mathcal{M}$. Theorem 11.1 produces $f : B^k \to \mathcal{EL}(S)$ extending g such that for $t \in B^k$ and $i \in \mathbb{N}, f_i(t)$ hits $\mathcal{M}$. Therefore, for each $t \in B^k, f(t)$ hits the closed marker $\bar{M}^*_N$ and hence $f(B^k) \subset \bar{U}_N \subset U'$. $\square$

13. $\mathcal{PML}(S)$ and $\mathcal{EL}(S)$ Approximation Lemmas

The main technical results of this paper are that under appropriate circumstances any map into $\mathcal{EL}(S)$ can be closely approximated by a map into $\mathcal{PML}(S)$ and vice versa. In this section we isolate out these results.

We first give a mild extension of Lemma 2.18, which is about approximating a map into $\mathcal{EL}(S)$ by a map into $\mathcal{PML}(S)$. The subsequent two results are about approximating maps into $\mathcal{PML}(S)$ by maps into $\mathcal{EL}(S)$.

Lemma 13.1. *Let K be a finite simplicial complex, $g : K \to \mathcal{EL}(S)$ and $\epsilon > 0$. Then there exists a generic PL map $h : K \to \mathcal{PML}(S)$ such that for each $t \in K$, $d_{PT(S)}(\phi(h(t)), g(t)) < \epsilon$ and $d_{\mathcal{PML}(S)}(h(t), \phi^{-1}(g(t'))) < \epsilon$ some $t' \in K$ with $d_K(t, t') < \epsilon$.*

Proof. Given g, Lemma 2.18 produces mappings $h_i : K \to \mathcal{PML}(S)$ such that $d_{PT(S)}(\phi(h_i(t)), g(t)) < \epsilon/i$. By super convergence we can assume that each h_i is a generic PL map satisfying the same conclusion.

We show that if i is sufficiently large, then the second conclusion holds for $h = h_i$. Otherwise, there exists a sequence $t_1, t_2, \cdots$ of points in K for which it fails respectively for $h_1, h_2, \cdots$. Since for each i, $h_i^{-1}(\mathcal{FPML}(S))$ is dense in K, it follows from Lemma 1.12 that we can replace the t_i's by another such sequence satisfying $d_{\mathcal{PML}(S)}(h_i(t_i), \phi^{-1}(g(t_i))) > \epsilon/2$, $t_i \to t_\infty$, $h_i(t_i) \to x_\infty$ and $\phi(h_i(t_i)) \in \mathcal{EL}(S)$ all $i \in \mathbb{N}$. By Lemmas 4.6 and 4.1, $\phi(h_i(t_i)) \to g(t_\infty)$ and hence by Lemma 1.12, $\phi(x_\infty) = g(t_\infty)$. Therefore, $\lim_{i\to\infty} d_{\mathcal{PML}(S)}(h_i(t_i), \phi^{-1}(g(t_\infty))) \leq \lim_{i\to\infty} d_{\mathcal{PML}(S)}(h_i(t_i), x_\infty) = 0$, a contradiction. $\square$

Lemma 13.2. *Let V be the underlying space of a finite simplicial complex. Let W be that of a finite subcomplex. S be a finite type hyperbolic surface with* $\dim(\mathcal{PML}(S)) = 2n+1$ *and let $\epsilon > 0$. If $h : V \to \mathcal{PMLEL}(S)$ is a generic PL map, $h^{-1}(\mathcal{EL}(S)) = W$ and* $\dim(V) \leq n$, *then there exists $g : V \to \mathcal{EL}(S)$ such that $g|W = h$ and for each $t \in V$, $d_{PT(S)}(g(t), \phi(h(t))) < \epsilon$.*

Proof. Since $k \leq n$, $\mathcal{A}(\phi(h(t))) \neq \emptyset$ for every $t \in V$, where $\mathcal{A}(\phi(h(t)))$ denotes the arational sublamination of $\phi(h(t))$. Therefore, by Lemma 3.5 for each $t \in V$, there exists an ϵ-marker $\mathcal{M}_t$ that is hit by $\phi(h(t))$. By Lemma 3.7, $\mathcal{M}_t$ is hit by all $\phi(h(s))$ for all s sufficiently close to t. By compactness of V, there exists an ϵ-marker family $\mathcal{J}$ hit by h. By Theorem 11.1 there exists $g : V \to \mathcal{EL}(S)$ such that g hits $\mathcal{J}$. Since both h and g hit $\mathcal{J}$, the conclusion follows. $\square$

Lemma 13.3. *S be a finite type hyperbolic surface with* $\dim(\mathcal{PML}(S)) = 2n + 1$. *Let V be the underlying space of a finite simplicial complex with* $\dim(V) \leq n$. *For $i \in \mathbb{N}$, let $\epsilon_i > 0$ and $h_i : V \to \mathcal{PML}(S)$ be generic PL maps. If for each $z \in \mathcal{EL}(S)$ there exists $\delta_z > 0$ so that for i sufficiently large $d_{\mathcal{PML}(S)}(h_i(V), \phi^{-1}(z)) > \delta_z$, then there exists $g_1, g_2, \cdots : V \to \mathcal{EL}(S)$ such that*

i) For each $i \in \mathbb{N}$ and $t \in V, d_{PT(S)}(g_i(t), \phi(h_i(t))) < \epsilon_i$.

ii) For each $z \in \mathcal{EL}(S)$, there exists a neighborhood $U_z \subset \mathcal{EL}(S)$ of z such that for i sufficiently large $g_i(V) \cap U_z = \emptyset$.

Proof. Given $z \in \mathcal{EL}(S)$ and $\delta_z > 0$, it follows from Lemma 4.2 that there exists $\epsilon(z) > 0$ such that if $x \in \mathcal{PML}(S)$ and $d_{PT(S)}(\phi(x), z') \le \epsilon(z)$, then $d_{\mathcal{PML}(S)}(x, \phi^{-1}(z)) < \delta_z$ and if $y \in \mathcal{EL}(S)$ and $d_{PT(S)}(y, z') \le \epsilon(z)$, then $\phi^{-1}(y) \subset N_{\mathcal{PML}(S)}(\phi^{-1}(z), \delta_z)$. Here z' is a diagonal extension of z. Let U_z be a neighborhood of z such that $y \in U_z$ implies $y \subset N_{PT(S)}(z', \epsilon(z)/2)$ for some diagonal extension z' of z.

Now let $f : V \to \mathcal{PML}(S)$ be a generic PL map with $k \le n$. If $0 < \epsilon \le \epsilon(z)$ and $d_{\mathcal{PML}(S)}(f(V),\, \phi^{-1}(z)) > \delta_z$, then we now show that there exists $g : V \to \mathcal{EL}(S)$ such that for every $t \in V$, $d_{PT(S)}(g(t), \phi(f(t))) < \epsilon$ and $g(V) \cap U_z = \emptyset$. By definition of $\epsilon(z)$, for every $t \in V$ and diagonal extension z' of z, $d_{PT(S)}(\phi(f(t)), z') > \epsilon(z)$ and in particular $d_{PT(S)}(A(t), z') > \epsilon$, where $A(t)$ is the arational sublamination of $\phi(f(t))$. By Lemma 3.4 for each $t \in V$, there exists an $\epsilon/2$ marker hit by $\phi(f(t))$. Since hitting markers is an open condition and V is compact there exists an $\epsilon/2$-marker family $\mathcal{J}$ hit by f. Now apply Theorem 11.1 to obtain $g : V \to \mathcal{EL}(S)$ such that g hits $\mathcal{J}$. Therefore for each $t \in V$, $d_{PT(S)}(g(t), \phi(f(t))) \le \epsilon/2$ and hence $g(t) \subsetneq N_{PT(S)}(z', \epsilon(z)/2)$ for all diagonal extensions z' of z. Therefore $g(t) \notin U_z$. It follows that $g(V) \cap U_z = \emptyset$.

Since $\mathcal{EL}(S)$ is separable and metrizable it is Lindelof, hence there exists a countable cover of $\mathcal{EL}(S)$ of the form $\{U_{z_j}\}$. By hypothesis for every $j \in \mathbb{N}$, there exists $n_j \in \mathbb{N}$ such that $i \ge n_j$ implies $d_{\mathcal{PML}(S)}(h_i(V), \phi^{-1}(z_j)) > \delta_{z_j}$. We can assume that $n_1 < n_2 < \cdots$. Let $m_i = \max\{j | n_j \le i\}$.

For $i \in \mathbb{N}$ apply the argument of the second paragraph with $f = h_i$ and $\epsilon = \min\{\epsilon(z_1), \cdots, \epsilon(z_{m_i}), \epsilon_i\}$ to produce $g_i : V \to \mathcal{EL}(S)$ satisfying for all $t \in V$, $d_{PT(S)}(g_i(t), \phi(h_i(t))) < \epsilon$ and such that $g_i(V) \cap U_{z_j} = \emptyset$ for $j \le m_i$, assuming that $m_i \ne \emptyset$. It follows that $g_1, g_2, \cdots$ satisfy the conclusion of the Lemma. ☐

14. Good cellulations of $\mathcal{PML}(S)$

The main result of this section, Proposition 14.6 produces a sequence of cellulations of $\mathcal{PML}(S)$ such that each cell is the polytope of a train trace and face relations among cells correspond to carrying among train tracks. Each cellulation is a subdivision of the previous one, subdivision of cells corresponds to splitting of train tracks and every train track eventually gets split arbitrarily much.

Definition 14.1. If τ is a train track, then let $V(\tau)$ denote the set of measured laminations carried by τ and $P(\tau)$ denote the polyhedron that is the quotient of $V(\tau)\backslash 0$, by rescaling. A train track is *generic* if exactly three edges locally emanate from each switch. All train tracks in this section are generic. We say that τ_2 is obtained from τ_1 by a *single splitting* if τ_2 is obtained by splitting without collision along a single large branch of τ_1. Also τ' is a *full splitting* of τ if it is the result of a sequence of single splittings along each large branch of τ.

Remark 14.2. By elementary linear algebra if τ_R and τ_L are the train tracks obtained from τ by a single splitting, then $V(\tau_R) = V(\tau)$ or $V(\tau_L) = V(\tau)$ or $V(\tau_R)$, $V(\tau_L)$ are obtained by slicing $V(\tau)$ along a codimension-1 plane through the origin.

Definition 14.3. We say that a finite set $R(\tau)$ of train tracks is *descended* from τ, if there exists a sequence of sets of train tracks $R_1 = \{\tau\}, R_2, \cdots, R_k = R(\tau)$ such that R_{i+1} is obtained from R_i by deleting one train track $\tau \in R_i$ and replacing it either by the two train tracks resulting from a single splitting of τ if $P(\tau)$ is split, or one of the resulting tracks with polyhedron equal to $P(\tau)$ otherwise. Finally, if a replacement track is not recurrent, then replace it by its maximal recurrent subtrack.

Remark 14.4. If $R(\tau)$ is descended from τ, then $P(\tau) = \cup_{\kappa \in R(\tau)} P(\kappa)$, each $P(\kappa)$ is codimension-0 in $P(\tau)$ and any two distinct $P(\kappa)$'s have pairwise disjoint interiors. Thus, any set of tracks descended from τ gives rise to a subdivision of τ's polyhedron. A consequence of Proposition 14.6 is that after subdivision, the lower dimensional faces of the subdivided $P(\tau)$ are also polyhedra of train tracks.

Definition 14.5. Let Δ be a cellulation of $\mathcal{PML}(S)$ and T a finite set of birecurrent generic train tracks. We say that T is *associated* to Δ if there exists a bijection between elements of T and cells of Δ such that if $\sigma \in \Delta$ corresponds to $\tau_\sigma \in T$, then $P(\tau_\sigma) = \sigma$.

Proposition 14.6. *Let S be a finite type hyperbolic surface. There exists a sequence of cellulations $\Delta_0, \Delta_1, \cdots$ of $\mathcal{PML}(S)$ such that*

i) Each Δ_{i+1} is a subdivision of Δ_i.

ii) Each Δ_i is associated to a set T_i of train tracks.

iii) If σ_j is a face of σ_k, then τ_{σ_j} is carried by τ_{σ_k}. If $\sigma_p \in \Delta_i, \sigma_q \in \Delta_j, \sigma_p \subset \sigma_q, \dim(\sigma_p) = \dim(\sigma_q)$ and $i > j$, then τ_{σ_p} is obtained from τ_{σ_q} by finitely many single splittings and possibly deleting some branches.

iv) There exists a subsequence $\Delta_{i_0} = \Delta_0, \Delta_{i_1}, \Delta_{i_2}, \cdots$ such that each complete $\tau \in T_{i_N}$ is obtained from a complete $\tau' \in T_0$ by N full splittings.

Definition 14.7. A sequence of cellulations $\{\Delta_i\}$ satisfying the conclusions of Proposition 14.6 is called a *good cellulation sequence*.

Proof. Let T_0 be the set of standard train tracks associated to a parametrized pants decomposition of S. As detailed in [PH], T_0 gives rise to a cellulation Δ_0 of $\mathcal{PML}(S)$ such that each $\sigma \in \Delta_0$ is the polyhedron of a track in T_0.

The idea of the proof is this. Suppose we have constructed $\Delta_0, \Delta_1, \cdots, \Delta_i$, and $T_0, T_1, \cdots T_i$ satisfying conditions i)-iii). Since any full splitting of a train track is the result of finitely many single splittings it suffices to construct Δ_{i+1} and T_{i+1} satisfying i)-iii) such that the complete tracks of T_{i+1} consist of the complete tracks of T_i, except that a single complete track τ of T_i is replaced by τ_R and/or τ_L. Let T_i' denote this new set of complete tracks. The key technical Lemma 14.8 implies that if σ is a codimension-1 cell of Δ_i with associated train track τ_σ, then every element of some set of train tracks descended from τ_σ is carried by an element of T_i'. This implies that σ has been subdivided in a manner compatible with the subdivision of the top dimensional cell $P(\tau)$. Actually, we must be concerned with lower dimensional cells too and new cells that result from any subdivision. Careful bookkeeping together with repeated applications of Lemma 14.8 deals with this issue.

Lemma 14.8. *Suppose that κ is carried by τ. Let τ_R and τ_L be the train tracks obtained from a single splitting of τ along the large branch b. Then there exists a set $R(\kappa)$ of train tracks descended from κ such that for each $\kappa' \in R(\kappa)$, κ' is carried by one of τ_R or τ_L.*

If $P(\tau) = P(\tau_R)$ (resp. $P(\tau_L)$), then each κ' is carried by the maximal recurrent subtrack of τ_R (resp. τ_L).

Proof. Let $N(\tau)$ be a fibered neighborhood of τ. Being carried by τ, we can assume that $\kappa \subset \mathrm{int}(N(\tau))$ and is transverse to the ties. Let $J \times [0,1]$ denote $\pi^{-1}(b)$ where $\pi : N(\tau) \to \tau$ is the projection contracting each tie to a point. Here $J = [0,1]$ and each $J \times t$ is a tie. Let $\partial_s(J \times i)$, $i \in \{0,1\}$ denote the singular point of $N(\tau)$ in $J \times i$. The canonical projection $h : J \times [0,1]$ to $[0,1]$ gives a *height* function on $J \times I$.

We can assume that distinct switches of κ inside of $J \times I$ occur at distinct heights and no switch occurs at heights 0 or 1. Call a switch s a *down* (resp. *up*) switch if two branches emanate from s that lie below (resp. above) s. A down (resp. up) switch is a *bottom* (resp. *top*) switch if the branches emanating below (resp. above) s extend to smooth arcs in $\kappa \cap J \times I$ that intersect $J \times 0 \setminus \partial_s(J \times 0)$ (resp. $J \times 1 \setminus \partial_s(J \times 1)$) in distinct components. Furthermore $h(s)$ is minimal (resp. maximal) with that property. There is at most one top switch and one bottom switch. Let s_T (resp. s_B) denote the top (resp. bottom) switch if it exists.

We define the, possibly empty, *b-core* as the unique embedded arc in $\kappa \cap J \times I$ transverse to the ties with endpoints in the top and bottom switches. We also require that $h(s_B) < h(s_T)$. Uniqueness follows since κ has no bigons.

We now show that if the b-core is empty, then κ is carried by one of τ_R or τ_L. To do this it suffices to show that after normal isotopy, κ has no switches in $J \times I$. By *normal isotopy* we mean isotopy of κ within $N(\tau)$ through train tracks that are transverse to the ties. It is routine to remove, via normal isotopy, the switches in $J \times I$ lying above s_T and those lying below s_B. Thus all the switches can be normally isotoped out of $J \times I$ if either s_T or s_B do not exist or $h(s_B) > h(s_T)$. If s_T exists, then since the *b*-core $= \emptyset$ all smooth arcs descending from s_T hit only one component of $J \times 0 \setminus \partial_s(J \times 0)$. Use this fact to first normally isotope κ to remove from $J \times I$ all the switches lying on smooth arcs from s_T to $J \times 0$ and then to isotope s_T out of $J \times I$.

Now assume that the *b*-core exists. Let $u_1, \cdots, u_r$, $d_1, \cdots, d_s$ be respectively the up and down switches of κ that lie on the core. For $i \in \{1, \cdots, s\}$, let $u(i)$ be the number of up switches in the b-core that lie above the down switch d_i. Define $C(\kappa) = \sum_{j=1}^{s} u(i)$. Define $C(\kappa) = 0$ if the *b*-core is empty. Note that if $C(\kappa) = 1$, then $r = s = 1$ and $u_1 = s_T$ and $d_1 = s_B$. Furthermore splitting κ along the large branch between s_T and s_B gives rise to tracks whose b-cores are empty and hence are carried by one of τ_R or τ_S.

Assume by induction that the first part of the lemma holds for all train tracks κ with $C(\kappa) < k$. Let κ be a track with $C(\kappa) = k$. Let e be a large branch of κ lying in κ's b-core. The two train tracks obtained by splitting along e have reduced C-values. Therefore, the first part of the lemma follows by induction.

Now suppose that $P(\tau) = P(\tau_R)$. It follows that $P(\tau_L) \subset P(\tau_R)$ and hence $P(\tau_L) = P(\tau_C)$, where τ_C is the train track obtained by splitting τ along b with collision. Therefore if κ' is the result of finitely many simple splittings, and is carried by τ_L, then the maximal recurrent subtrack of κ' is carried by τ_C and hence by the maximal recurrent subtrack of τ_R. $\square$

Remarks 14.9. i) If each u_i is to the *left* of the b-core and each d_j is to the *right* of the b-core, then if κ_R and κ_L are the tracks resulting from a single splitting along a large branch in the b-core, then $C(\kappa_R) = C(\kappa) - 1$, while $C(\kappa_L) = 0$.

ii) Any large branch of κ that intersects the b-core is contained in the b-core. It follows that if κ' is the result of finitely many single splittings of κ, then $C(\kappa') \leq C(\kappa)$.

Now assume we have a sequence $(\Delta_0, T_0), \cdots, (\Delta_i, T_i)$ of cellulations and associated train tracks that satisfy i)-iii) of the proposition. Let σ be a cell of Δ_i with associated train track τ. Since any full splitting of a track is the concatenation of splittings along large branches, to complete the proof of the proposition, it suffices to prove the following claim.

Claim. *If τ_R and τ_L are the result of a single splitting of τ, then there exists a cellulation Δ_{i+1} with associated train tracks T_{i+1} extending the sequence and satisfying i)-iii), so that τ is replaced with the maximal recurrent subtracks of τ_R and/or τ_L and if $\tau' \in T_i$ is such that $\dim(P(\tau')) \geq \dim(P(\tau))$ and $\tau' \neq \tau$, then $\tau' \in T_{i+1}$.*

Proof. Proof by induction on $\dim(\sigma)$. We will assume that each of $P(\tau_R)$ and $P(\tau_L)$ are proper subcells of $P(\tau)$, for proof in the general case is similar. The claim is trivial if $\dim(\sigma) = 0$. Now assume that the claim is true if $\dim(\sigma) < k$. Assuming that $\dim(\sigma) = k$ let $\sigma_1, \cdots, \sigma_p$, be the $(k-1)$-dimensional faces of σ with corresponding train tracks $\kappa_1, \cdots, \kappa_p$. By Lemma 14.8 there exists sets $R(\kappa_1), \cdots, R(\kappa_p)$ descended from the κ_i's such that any train track in any of these sets is carried by one of τ_R and τ_L.

If $R(\kappa_1) \neq \{\kappa_1\}$, then there exists a single splitting of κ_1 into κ_1^1 and κ_1^2 such that $R(\kappa_1) = R(\kappa_1^1) \cup R(\kappa_1^2)$, where $R(\kappa_1^i)$ is descended from κ_1^i. (As usual, only one may be relevant and it might have stuff deleted.) By induction there exists a cellulation Δ_i^1 with associated T_i^1 such that the sequence $(\Delta_0, T_0), \cdots, (\Delta_i, T_i)$, (Δ_i^1, T_i^1) satisfies i)-iii) and in the passage from T_i to T_i^1, $P(\kappa_1)$ is the only polyhedron of dimension $\geq k-1$ that gets subdivided and it is replaced by $P(\kappa_1^1)$ and $P(\kappa_1^2)$. By repeatedly applying the induction hypothesis we obtain the sequence $(\Delta_0, T_0), \cdots, (\Delta_i, T_i), (\Delta_i^1, T_i^1), \cdots, (\Delta_i^p, T_i^p)$ satisfying i)-iii) such that in the passage from Δ_i to Δ_i^p, each σ_j is subdivided into the polyhedra of $R(\kappa_j)$ and no cells of dimension $\geq k$ are subdivided and their associated train tracks are unchanged. It follows that if Δ_{i+1} is obtained by subdividing Δ_i^p by replacing σ by $P(\tau_R)$, $P(\tau_L)$ and $P(\tau_C)$ and T_{i+1} is the set of associated train tracks, then $(\Delta_1, T_1), \cdots, (\Delta_i, T_i), (\Delta_{i+1}, T_{i+1})$ satisfies i)-iii) and hence the induction step is completed. $\square$

Lemma 14.10. *Let $\Delta_1, \Delta_2, \cdots$ be a good cellulation sequence.*

i) If σ is a cell of Δ_i, then $\phi(\sigma) \cap \mathcal{EL}(S)$ is closed in $\mathcal{EL}(S)$.

ii) If $\mathcal{L} = \phi(x)$ some $x \in \mathcal{PML}(S)$, then for each $i \in \mathbb{N}$ there exists a unique cell $\sigma \in \Delta_i$ such that $\phi^{-1}(\mathcal{L}) \subset \operatorname{int}(\sigma)$.

iii) If U is an open set of $\mathcal{PML}(S)$ that is the union of open cells of Δ_i, then $\phi(U) \cap \mathcal{EL}(S)$ is open in $\mathcal{EL}(S)$.

iv) If σ is a cell of Δ_i, then $\operatorname{int}(\sigma)$ is open in $\phi(\sigma) \cap \mathcal{EL}(S)$.

v) If $\mu \in \mathcal{EL}(S)$ and $\phi^{-1}(\mu) \in \operatorname{int}(\sigma)$, σ a cell of Δ_i, then there exists $\epsilon(\mu) > 0$ such that $B(\mu, \epsilon(\mu)) = \{x \in \mathcal{PML}(S) | d_{PT(S)}(\phi(x), \mu) < \epsilon(\mu)\} \subset \hat{\sigma}$.

Proof. Conclusion i) follows from Lemma 1.14.

Conclusion ii) follows from the fact that if τ carries $\mathcal{L}$, then $\phi^{-1}(\mathcal{L}) \subset P(\tau)$.

Conclusion iii) follows from Conclusion i) and the fact that $\phi(U) \cap \phi(\mathcal{PML}(S) \setminus U) \cap \mathcal{EL}(S) = \emptyset$.

Conclusion iv) follows from Conclusion i) and the fact that $\phi(\text{int}(\sigma)) \cap \phi(\partial\sigma) \cap \mathcal{EL}(S) = \emptyset$.

Let κ be a cell of Δ_i with associated train track τ. If τ carries μ, then $\kappa \cap \sigma \neq \emptyset$ and hence by ii) σ is a face of κ and hence $\text{int}(\kappa) \subset \hat{\sigma}$. If τ does not carry μ, then by Lemma 1.16 $B(\mu, \epsilon) \cap P(\tau) = \emptyset$ for ϵ sufficiently small. Since Δ_i has finitely many cells, the result follows. □

We have the following PML-approximation result for good cellulation sequences.

Proposition 14.11. *Let $\Delta_1, \Delta_2, \cdots$ be a good cellulation sequence, K a finite simplicial complex, $i \in \mathbb{N}$ fixed and $g : K \to \mathcal{EL}(S)$. Then there exists $h : K \to \mathcal{PML}(S)$ a generic PL map such that for each $t \in K$, there exists $\sigma(t)$ a cell of Δ_i such that $h(t) \cup \phi^{-1}(g(t)) \subset \hat{\sigma}(t)$.*

Proof. Given $\mu \in \mathcal{EL}(S)$, let σ be the cell of Δ_i such that $\phi^{-1}(\mu) \subset \text{int}(\sigma)$. Let $\epsilon(\mu)$ be as in Lemma 14.10 v). By Lemmas 4.6 and 4.1 there exists a neighborhood $U_\mu \subset \mathcal{EL}(S)$ of μ and $\delta_\mu > 0$ such that $d_{PT(S)}(\mathcal{L}, z) < \delta_\mu, z \in U_\mu, \mathcal{L} \in \mathcal{L}(S)$ implies that $d_{PT(S)}(\mathcal{L}, \mu) < \epsilon(\mu)$.

The compactness of $g(K)$ implies that there exists a finite open cover $U_{\mu_1}, \cdots, U_{\mu_k}$ of $g(K)$. Let $\delta = \min\{\delta_{\mu_1}, \cdots, \delta_{\mu_k}\}$. Then any δ PML-approximation as in Proposition 13.1 satisfies the conclusion of the proposition. □

Lemma 14.12. *If $\sigma = P(\tau)$ is a cell of Δ_i, σ' is a face of σ and $\mu \in \phi(\sigma') \cap \mathcal{EL}(S)$, then τ fully carries a diagonal extension of μ.*

Proof. Apply Lemma 4.8 and Proposition 14.6 iii). □

Lemma 14.13. *Let $\Delta_1, \Delta_2, \cdots$ be a good cellulation sequence. Given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $i \geq N$, $\mathcal{L}_1 \in \mathcal{EL}(S)$, $\mathcal{L}_2 \in \mathcal{L}(S)$ and $\mathcal{L}_1, \mathcal{L}_2$ are carried by τ some $\tau \in T_i, i \geq N$, then $d_{PT(S)}(\mathcal{L}_1, \mathcal{L}_2) < \epsilon$.*

Proof. Apply Propositions 14.6 and 4.10. □

Definition 14.14. If σ is a cell of Δ_i, then define the *open star* $\hat{\sigma} = \cup_{\eta \in \Delta_i | \sigma \subset \eta} \text{int}(\eta)$.

Lemma 14.15. *If σ is a cell of Δ_i, then $\hat{\sigma}$ is open in $\mathcal{PML}(S)$ and contractible. Indeed, it deformation retracts to* $\text{int}(\sigma)$. □

Lemma 14.16. *If σ is a face of κ, then $\hat{\kappa} \subset \hat{\sigma}$.* □

Definition 14.17. Define $U(\sigma) = \phi(\hat{\sigma}) \cap \mathcal{EL}(S)$.

Lemma 14.18. *$\mathcal{B} = \{U(\sigma) | \phi(\text{int}(\sigma)) \cap \mathcal{EL}(S) \neq \emptyset$ and $\sigma \in \Delta_i$ for some $i \in \mathbb{N}\}$ is a neighborhood basis of $\mathcal{EL}(S)$.*

Proof. By Lemma 4.1 it suffices to show that for each $\mu \in \mathcal{EL}(S)$ and $\epsilon > 0$, there exists a neighborhood $U(\sigma)$ of μ such that $z \in U(\sigma)$ implies $d_{PT(S)}(z, \mu) < \epsilon$. Choose N such that the conclusion of Lemma 14.13 holds. Let σ be the simplex of Δ_N such that $\phi^{-1}(\mu) \subset \text{int}(\sigma)$. If $z \in U(\sigma)$, then $\phi^{-1}(z) \in P(\tau)$ where σ is a face of τ. Therefore, z and μ are carried by τ and hence $d_{PT(S)}(\mu, z) < \epsilon$. □

Lemma 14.19. *Let $\mathcal{U}$ be an open cover of $\mathcal{EL}(S)$. Then there exists a refinement $\mathcal{U}_2$ of $\mathcal{U}$ by elements of $\mathcal{B}$, maximal with respect to inclusion, such that for each $U(\sigma_2) \in \mathcal{U}_2$ there exists $U \in \mathcal{U}$ and $\delta(U(\sigma_2)) > 0$ with the following property. If $x \in \hat{\sigma}_2$ and $z \in \mathcal{EL}(S)$ are such that $d_{PT(S)}(\phi(x), z) < \delta(U(\sigma_2))$, then $z \in U$.*

Proof. It suffices to show that if $\mu \in \mathcal{EL}(S)$ and $U \in \mathcal{U}$, then there exists σ_2 and $\delta(\sigma_2)$ such that the last sentence of the lemma holds. By Lemma 4.1 there exists $\delta_1 > 0$ such that if $z \in \mathcal{EL}(S)$ and $d_{PT(S)}(z, \mu) < \delta_1$, then $z \in U$. By Proposition 4.10 there exists $N_1 > 0$ and $\delta > 0$ such that if τ is obtained by fully splitting any one of the train tracks associated to the top dimensional cells of Δ_1 at least N_1 times, μ is carried by $\tau, \mathcal{L} \in \mathcal{L}(S)$ is carried by $\tau, z \in \mathcal{EL}(S)$ with $d_{PT(S)}(z, \mathcal{L}) < \delta$, then $d_{PT(S)}(z, \mu) < \delta_1$ and hence $z \in U$. Therefore, if σ is the cell of Δ_{N_1} such that $\mathrm{int}(\sigma)$ contains $\phi^{-1}(\mu)$, then let $\sigma_2 = \sigma$ and $\delta(U(\sigma_2)) = \delta$. $\square$

15. Bounds on the dimension of $\mathcal{EL}(S)$

In this section we give an upper bound for $\dim(\mathcal{EL}(S))$ for any finite type hyperbolic surface S. When S is either a punctured sphere or torus we give lower bounds for $\dim(\mathcal{EL}(S))$. For punctured spheres these bounds coincide. We conclude that if S is the $n+4$ punctured sphere, then $\dim(\mathcal{EL}(S)) = n$.

To compute the lower bounds we will show that $\pi_n(\mathcal{EL}(S)) \neq 0$ for $S = S_{0,4+n}$ or $S_{1,1+n}$. Since $\mathcal{EL}(S)$ is $(n-1)$-connected and $(n-1)$-locally connected Lemma 15.4 applies.

To start with, $\mathcal{EL}(S)$ is a separable metric space and hence covering dimension, inductive dimension and cohomological dimension of $\mathcal{EL}(S)$ all coincide [HW]. By *dimension* we mean any of these equal values.

The next lemma is key to establishing the upper bound for $\dim(\mathcal{EL}(S))$.

Lemma 15.1. *Let $S = S_{g,p}$, where $p > 0$ and let $\{\Delta_i\}$ be a good cellulation sequence. If σ is a cell of Δ_i and $\phi(\sigma) \cap \mathcal{EL}(S) \neq \emptyset$, then* $\dim(\sigma) \geq n + 1 - g$*, where* $\dim(\mathcal{PML}(S)) = 2n + 1$.

If $S = S_{g,0}$ and σ is as above, then $\dim(\sigma) \geq n + 2 - g$.

Proof. We first consider the case $p \neq 0$. By Proposition 14.6, $\sigma = P(\tau)$ for some train track τ. It suffices to consider the case that τ is generic. A generic track with e edges has $2e/3$ switches, hence $\dim(V(\tau)) \geq e - 2e/3 = e/3$, e.g. see page 116 [PH]. Since τ carries an element of $\mathcal{EL}(S)$, all complementary regions are discs with at most one puncture. After filling in the punctures, τ has say f complementary regions all of which are discs, thus $2 - 2g = \chi(S_g) = 2e/3 - e + f$ and hence $\dim(V(\tau)) \geq e/3 = f - 2 + 2g$. Therefore, $\dim(P(\tau)) = \dim(V(\tau)) - 1 \geq f - 3 + 2g \geq p - 3 + 2g = n + 1 - g$, since $2n + 1 = 6g + 2p - 7$.

If $p = 0$, then $f \geq 1$ and hence the above argument shows $\dim(P(\tau)) = \dim(V(\tau)) - 1 \geq f - 3 + 2g \geq 1 - 3 + 2g = n + 2 - g$. $\square$

Corollary 15.2. *If $S = S_{g,p}$ with $p > 0$ (resp. $p = 0$), then for each $m \in \mathbb{Z}$, $\mathcal{EL}(S) = U(\sigma_1) \cup \cdots \cup U(\sigma_k)$, where $\{\sigma_1, \cdots, \sigma_k\}$ are the cells of Δ_m of dimension $\geq n + 1 - g$ (resp. $\geq n + 2 - g$).* $\square$

Proposition 15.3. *Let $S = S_{g,p}$. If $p > 0$ (resp. $p = 0$), then* $\dim(\mathcal{EL}(S)) \leq 4g + p - 4 = n + g$ *(resp.* $\dim(\mathcal{EL}(S)) \leq 4g - 5 = n + g - 1$*), where* $\dim(\mathcal{PML}(S)) = 2n + 1$.

Proof. To minimize notation we give the proof for the $g = 0$ case. The general case follows similarly, after appropriately shifting dimensions and indices.

Let $\mathcal{E}_r$ denote those elements of $\mu \in \mathcal{EL}(S)$ such that $\mu = \phi(x)$ for some x in the r-skeleton Δ_i^r of some Δ_i. We will show that for each i, $\dim(\mathcal{E}_i) \leq \max\{i - (n + 1), -1\}$. We use the convention that $\dim(X) = -1$ if $X = \emptyset$.

By Lemma 15.1 $\Delta_i^r \cap \mathcal{EL}(S) = \emptyset$ for all $i \in \mathbb{N}$ and $r \leq n$ and hence $\mathcal{E}_n = \emptyset$. Now suppose by induction that for all $k < m$, $\dim(\mathcal{E}_{n+1+k}) \leq k$. We will show that $\dim(\mathcal{E}_{n+1+m}) \leq m$. Now $\mathcal{E}_{n+1+m} = \cup_{i\in\mathbb{N}}\phi(\Delta_i^{n+1+m}) \cap \mathcal{EL}(S)$, each $\phi(\Delta_i^{n+1+m}) \cap \mathcal{EL}(S)$ is closed in $\mathcal{EL}(S)$ by Lemma 1.14 and $\mathcal{EL}(S)$ is separable metric, hence to prove that $\dim(\mathcal{E}_{n+1+m}) \leq m$ it suffices to show by the sum theorem (Theorem III 2 [HW]) that $\dim(\phi(\Delta_i^{n+1+m}) \cap \mathcal{EL}(S)) \leq m$ for all $i \in \mathbb{N}$.

Let $X = \phi(\Delta_i^{n+1+m}) \cap \mathcal{EL}(S)$. To show that the inductive dimension of X is $\leq m$ it suffices to show that if U is open in X, $\mu \in U$, then there exists V open in X with $\mu \in V$ and $\partial V \subset \mathcal{E}_{n+m}$.

Let $\sigma \in \Delta_j$, some $j \geq i$, such that $\mu \in U(\sigma) \cap X \subset U$. Such a σ exists by Lemma 14.18. Now $\hat{\sigma} = \cup_{u\in J}\operatorname{int}(\sigma_u)$ where the union is over all cells in Δ_j having σ as a face. Since $\Delta_i^{n+1+m} \subset \Delta_j^{n+1+m}$ it follows that after reindexing, $U(\sigma) \cap X \subset \phi(\operatorname{int}(\sigma_1) \cup \cdots \cup \operatorname{int}(\sigma_q)) \cap \mathcal{EL}(S)$, where $\sigma_1, \cdots, \sigma_q$ are those σ_u's such that $\dim(\sigma_i) \leq n+m+1$.

By Lemma 1.14 $(\cup_{i=1}^q \phi(\sigma_i)) \cap \mathcal{EL}(S)$ is closed in $\mathcal{EL}(S)$ and hence restricts to a closed set in X. It follows that $\partial(U(\sigma) \cap X) \subset (\cup_{i=1}^q \phi(\partial\sigma_i)) \cap \mathcal{EL}(S) \subset \mathcal{E}_{n+m}$. ☐

To establish our lower bounds on $\dim(\mathcal{EL}(S))$ we will use the following basic result.

Lemma 15.4. *Let X be a separable metric space such that X is $(n-1)$-connected and $(n-1)$-locally connected. If $\pi_n(X) \neq 0$, then $\dim(X) \geq n$.*

Proof. (Dranishnikov [Dr]) Let $f : S^n \to X$ be an essential map. If $\dim(X) \leq n-1$, then by [HW] $\dim(f(S^n)) \leq \dim(X)$ and hence $\dim(f(X))$ is at most $n-1$. By Bothe [Bo] there exists a compact, metric, absolute retract Y such $\dim(Y) \leq n$ and $f(S^n)$ embeds in Y. By Theorem 10.1 [Hu], since Y is metric and X is $(n-1)$-connected and $(n-1)$-locally connected, the inclusion of $f(S^n)$ into X extends to a map $g : Y \to X$. Now Y is contractible since it is an absolute retract. (The cone of Y retracts to Y.) It follows that f is homotopically trivial, which is a contradiction. ☐

We need the following controlled homotopy lemma for $\mathcal{PML}(S)$ approximations that are very close to a given map into $\mathcal{EL}(S)$.

Lemma 15.5. *Let K be a finite simplicial complex. Let $g : K \to \mathcal{EL}(S)$. Let $U \subset \mathcal{PML}(S)$ be a neighborhood of $\phi^{-1}(g(K))$. There exists $\delta > 0$ such that if $f_0, f_1 : K \to \mathcal{PML}(S)$ and for every $t \in K$ and $i \in \{0,1\}$, $d_{PT(S)}(\phi(f_i(t)), g(t)) < \delta$, then there exists a homotopy from f_0 to f_1 supported in U.*

Proof. Let $\Delta_1, \Delta_2, \cdots$ be a good cellulation sequence of $\mathcal{PML}(S)$. Given $x \in \mathcal{EL}(S)$ and $i \in \mathbb{N}$ let σ_x^i denote the cell of Δ_i such that $\phi^{-1}(x) \subset \operatorname{int}(\sigma_x^i)$. Such a σ_x^i exists since each cell of Δ_i is associated to a train track. By Lemma 4.4, given $x \in \mathcal{EL}(S)$ there exists a neighborhood W_x of x such that $\phi^{-1}(W_x) \subset U$ and lies in a small neighborhood of the cell $\phi^{-1}(x)$, i.e. that only intersects σ_x^i and higher dimensional cells that have σ_x^i as a face. Therefore, $y \in W_x$ implies that σ_x^i is a face of σ_y^i and hence $\hat{\sigma}_y^i \subset \hat{\sigma}_x^i$.

Let U' be a neighborhood of $\phi^{-1}(g(K))$ such that $\bar{U}' \subset U$. By Lemmas 14.18 and 4.4, for i sufficiently large, $\phi^{-1}(U(\sigma_x^i)) \subset U'$. This implies that $\hat{\sigma}_x^i \subset U$, since $\mathcal{FPML}(S)$ is dense in $\mathcal{PML}(S)$. Let $U_i = \cup_{x\in g(K)}\hat{\sigma}_x^i$. By compactness of $g(K)$ and the previous paragraph it follows that for i sufficiently large $U_i \subset U$. Fix such an i. Let $\Sigma = \{\sigma \in \Delta_i |$ some σ_x^i is a face of $\sigma.\}$.

There exists $\delta > 0$ and a function $W : g(K) \to \Sigma$ such that if $y \in \mathcal{PML}(S), x \in g(K)$ and $d_{PT(S)}(x, \phi(y)) < \delta$, then $y \in \hat{W}(x)$. Furthermore, if $d_{PT(S)}(x_1, \phi(y)) < \delta, \cdots, d_{PT(S)}(x_u, \phi(y)) < \delta$, then after reindexing the x_i's, $W(x_1) \subset \cdots W(x_u)$. Modulo the epsilonics (to find δ) which follow from Lemmas 4.3 and 4.2 (i.e. super convergence), W is defined as follows. Define $d : g(K) \to \mathbb{N}$ by $d(x) = \dim(\sigma_x^i)$. Let p be the minimal value of $d(g(K))$. Define $W(x) = \sigma_x^i$ if $d(x) = p$. Next define $W(x') = \sigma_x^i$ if $d_{PT(S)}(x, x')$ is very small, where $d(x) = p$. To guarantee that the second sentence holds, we require that if both $d_{PT(S)}(x_1, x')$ and $d_{PT(S)}(x_2, x')$ are very small, then $W(x_1) = W(x_2)$. If $W(x)$ has not yet been defined and $d(x) = p + 1$, then define $W(x) = \sigma_x^i$. Next define $W(x') = \sigma_x^i$ if $W(x')$ hasn't already been defined and $d_{PT(S)}(x, x')$ is very very small, where $d(x) = p + 1$. Again we require that if $d_{PT(S)}(x_1, x')$ and $d_{PT(S)}(x_2, x')$ are very very small, then after reindexing $W(x_1) \subset W(x_2)$. Inductively, continue to define W on all of $g(K)$. Since $\dim(\Delta_i)$ is finite this process eventually terminates. Take δ to be the minimal value used to define smallness.

Next subdivide K such that the following holds. For every simplex κ of K, there exists $t \in \kappa$ such that for all $s \in \kappa$, $d_{PT(S)}(\phi(f_0(s)), g(t)) < \delta$ and $d_{PT(S)}(\phi(f_1(s)), g(t)) < \delta$. Thus by the previous paragraph, for each simplex κ of K, $f_0(\kappa) \cup f_1(\kappa) \subset \hat{W}(g(t))$ for some $t \in \kappa$ where t satisfies the above property. Let $\sigma(\kappa)$ denote the maximal dimensional simplex of Σ with these properties. Note that by the second sentence of the previous paragraph, $\sigma(\kappa)$ is well defined and if κ' is a face of κ, then $\sigma(\kappa)$ is a face of $\sigma(\kappa')$.

We now construct the homotopy $F : K \times I \to \mathcal{PML}(S)$ from f_0 to f_1. Assume that K has been subdivided as in the previous paragraph. If v is a vertex of K, then both $f_0(v), f_1(v) \in \hat{\sigma}(v)$ which is contractible by Lemma 14.15. Thus F extends over $v \times I$ such that $F(v \times I) \subset \hat{\sigma}(v)$. Assume by induction that if η is a simplex of K and $\dim(\eta) < m$, then F has been extended over $\eta \times I$ with $F(\eta) \subset \hat{\sigma}(\eta)$. If κ is an m-simplex, then the contractibility of $\hat{\sigma}(\kappa)$ enables us to extend F over $\kappa \times I$, with $F(\kappa \times I) \subset \hat{\sigma}(\kappa)$. □

Theorem 15.6. *If $S = S_{0,4+n}$ or $S_{1,1+n}$, then $\pi_n(\mathcal{EL}(S)) \neq 0$.*

Proof. Let S denote either $S_{0,4+n}$ or $S_{1,1+n}$. In either case $\dim(\mathcal{PML}(S)) = 2n+1$. By Harer [Hr] (see also Ivanov [Iv], [IJ]) the curve complex $\mathcal{C}(S)$ is homotopy equivalent to a non trivial wedge of n-spheres. Thus, there exists an essential map $h : S^n \to \mathcal{C}(S)$. We can assume that $h(S^n)$ is a simplicial map with image a finite subcomplex L of the n-skeleton of $\mathcal{C}(S)$ and hence $0 \neq [h_*([S^n])] \in H_n(L)$. We abuse notation by letting L denote the image of L under the natural embedding, given in Definition 1.20. Let Z be a simplicial n-cycle that represents $[h_*([S^n])]$. Let σ be a n-simplex of L in the support of Z. Let $f_0 : S^n \to \mathcal{PML}(S) \setminus L$ be a PL embedding which links σ, i.e. there exists an extension $F_0 : B^{n+1} \to \mathcal{PML}(S)$ of f_0 such that F_0 is transverse to L and intersects L at a single point in $\mathrm{int}(\sigma)$. It follows that $f_0[S^n]$ has non trivial linking number with Z and hence f_0 is homotopically non trivial as a map into $\mathcal{PML}(S) \setminus L$. The theorem is now a consequence of the following lemma. □

Lemma 15.7. *Let S be a finite type hyperbolic surface with $\dim(\mathcal{PML}(S)) = 2n + 1$. Let L be an n-dimensional subcomplex of $\mathcal{C}(S)$ that is naturally embedded in $\mathcal{PML}(S)$. If $\pi_n(\mathcal{PML}(S) \setminus L) \neq 0$, then $\pi_n(\mathcal{EL}(S)) \neq 0$. Indeed, if $f : S^n \to$*

$\mathcal{PML}(S) \setminus L$ is essential, then there exists a map $F : S^n \times I \to \mathcal{PMLEL}(S) \setminus L$ such that $F|S^n \times 0 = f$, $F(S^n \times [0,1)) \subset \mathcal{PML}(S) \setminus L$ and $0 \neq [F|S^n \times 1] \in \pi_n(\mathcal{EL}(S))$.

Proof. View f as a map into $\mathcal{PMLEL}(S)$ with image in $\mathcal{PML}(S)$.

Step 1. f extends to a map $f : S^n \times I \to \mathcal{PMLEL}(S)$ such that $f(S^n \times [0,1) \subset \mathcal{PML}(S) \setminus L$ and $f(S^n \times 1) \subset \mathcal{EL}(S)$.

Proof of Step 1. Let Σ be a triangulation of S^n. It suffices to consider the case that f is a generic PL map. Thus $f(\Sigma^0) \subset \mathcal{FPML}(S)$. Extend $f|\Sigma^0 \times 0$ to $\Sigma^0 \times I$ by $f(t,s) = f(t,0)$ where $f(t,1)$ is viewed as an element of $\mathcal{EL}(S)$. Now for some $u \leq n$, assume that f extends as desired to $\Sigma^u \times I \cup S^n \times 0$. If σ is a $(u+1)$-simplex of Σ, then $f|\sigma \times 0 \cup \partial\sigma \times I$ maps into $\mathcal{PMLEL}(S)$ with $f(\partial\sigma \times 1) \subset \mathcal{EL}(S)$ and the rest mapping into $\mathcal{PML}(S) \setminus L$. By Lemma 11.1 f extends to $\sigma \times I$ such that $f(\sigma \times 1) \subset \mathcal{EL}(S)$ and $f(\sigma \times [0,1)) \subset \mathcal{PML}(S) \setminus L$. Step 1 now follows by induction. □

Step 2 If $g = f|S^n \times 1$, then g is an essential map into $\mathcal{EL}(S)$.

Proof of Step 2. Otherwise there exists $G : B^{n+1} \to \mathcal{EL}(S)$ extending g. Since $L \subset \mathcal{C}(S)$, $L \cap \phi^{-1}(G(B^{n+1})) = \emptyset$. Let $F : B^{n+1} \to \mathcal{PML}(S)$ be a $\delta - \mathcal{PML}(S)$ approximation of G. By Lemma 13.1 $F(B^{n+1}) \cap L = \emptyset$ for δ sufficiently small. Now $F|S^n$ and $f|S^n \times s$ are respectively δ, δ'-$\mathcal{PML}(S)$ approximations of g, where $\delta' \to 0$ as $s \to 1$. Since δ can be chosen arbitrarily small it follows from Lemma 15.5 that F can be chosen such that for s sufficiently large, $f|S^n \times s$ and $F|S^n$ are homotopic via a homotopy supported in $\mathcal{PML}(S) \setminus L$. By concatenating this homotopy with F, we conclude that f is homotopically trivial via a homotopy supported in $\mathcal{PML}(S) \setminus L$; which is a contradiction. □

Proposition 15.8. *If $S = S_{0,p}$, then $\dim(\mathcal{EL}(S)) \geq p - 4$. If $S = S_{1,p}$, then $\dim(\mathcal{EL}(S)) \geq p-1$. In either case if $\dim(\mathcal{PML}(S)) = 2n+1$, then $\dim(\mathcal{EL}(S)) \geq n$.*

Proof. By Theorems 11.2 and 12.1, $\mathcal{EL}(S)$ is $(n-1)$-connected and $(n-1)$-locally connected. By Theorem 15.6, $\pi_n(\mathcal{EL}(S)) \neq 0$. Therefore by Lemma 15.4, $\dim(\mathcal{EL}(S)) \geq n$. □

By Proposition 15.8 and Proposition 15.3 we obtain;

Theorem 15.9. *If S is the $(4+n)$-punctured sphere, then $\dim(\mathcal{EL}(S)) = n$.* □

Theorem 15.10. *If S is the $(n+1)$-punctured torus, then $n+1 \geq \dim(\mathcal{EL}(S)) \geq n$.* □

Remark 15.11. In a future paper we will show that if $S = S_{g,p}$ with $p > 0$, then $\dim(\mathcal{EL}(S)) \leq n + g - 1$.

16. Nobeling spaces

The n-dimensional *Nobeling space* $\mathbb{R}^{2n+1}_n$ is the space of points in $\mathbb{R}^{2n+1}$ with at most n rational coordinates. The goal of the next two sections is to complete the proof of the following theorem.

Theorem 16.1. *If S is the $n+p$ punctured sphere, then $\mathcal{EL}(S)$ is homeomorphic to the n-dimensional Nobeling space.*

By Luo [Luo], $\mathcal{C}(S_{2,0})$ is homeomorphic to $\mathcal{C}(S_{0,6})$ and thus by Klarreich [K] $\mathcal{EL}(S_{2,0})$ is homeomorphic to $\mathcal{EL}(S_{0,6})$.

Corollary 16.2. *If S is the closed surface of genus-2, then $\mathcal{EL}(S)$ is homeomorphic to the Nobeling surface, i.e. the 2-dimensional Nobeling space.*

Remarks 16.3. Using [G1], Sabastian Hensel and Piotr Przytycki earlier proved that the ending lamination space of the 5-times punctured sphere is homeomorphic to the Nobeling curve. They used Luo [Luo] and Klarreich [K] to show that $\mathcal{EL}(S_{1,2})$ is also homeomorphic to the Nobeling curve.

Hensel and Przytycki boldly conjectured that if $\dim(\mathcal{PML}(S)) = 2n+1$, then $\mathcal{EL}(S)$ is homeomorphic to $\mathbb{R}^{2n+1}_n$. Theorem 16.1 gives a positive proof of their conjecture for punctured spheres.

Independently, in 2005, Ken Bromberg and Mladen Bestvina asked [BB] if ending lamination spaces are Nobeling spaces.

Remarks 16.4. Historically, the m-dimensional Nobeling space was called the *universal Nobeling space of dimension m* and a Nobeling space was one that is locally homeomorphic to the universal Nobeling space. In 2006, S. Ageev [Ag], Michael Levin [Le] and Andrzej Nagorko [Na] independently showed that any two connected Nobeling spaces of the same dimension are homeomorpic. The 0 and 1-dimensional versions of that result were respectively given in [AU] 1928 and [KLT] 1997.

These spaces were named after Georg Nobeling who showed [N] in 1930 that any m-dimensional separable metric space embeds in an m-dimensional Nobeling space. This generalized a result by Nobeling's mentor, Karl Menger, who defined [Me] in 1926 the Menger compacta and showed that any 1-dimensional compact metric space embeds in the Menger curve. A topological characterization of m-dimensional Menger compacta was given in [Be] by Mladen Bestvina in 1984.

The following equivalent form of the topological characterization of m-dimensional Nobeling spaces is due to Andrzej Nagorko.

Theorem 16.5. *(Nagorko [HP]) A topological space X is homeomorphic to the m-dimensional Nobeling space if and only if the following conditions hold.*

i) X is separable

ii) X supports a complete metric

iii) X is m-dimensional

iv) X is $(m-1)$-connected

v) X is $(m-1)$-locally connected

vi) X satisfies the locally finite m-discs property

Definition 16.6. [HP] The space X satisfies the *locally finite m-discs property* if for each open cover $\{\mathcal{U}\}$ of X and each sequence $f_i : B^m \to X$, there exists a sequence $g_i : B^m$ to X such that

i) for each $x \in X$ there exists a neighborhood U of x such that $g_i(B^m) \cap U = \emptyset$ for i sufficiently large.

ii) for each $t \in B^m$, there is a $U \in \mathcal{U}$ such that $f_i(t), g_i(t) \in U$.

Proof of Theorem 16.1. By Remark 1.6 $\mathcal{EL}(S)$ is separable and supports a complete metric. If $S = S_{0,n+4}$, then conditions iii)-v) of Theorem 16.5 respectively follow from Theorems 11.2, 12.1 and 15.9.

To complete the proof of Theorem 16.1 it suffices to show that $\mathcal{EL}(S_{0,n+4})$ satisfies the locally finite n-discs property.

17. The locally finite n-discs property

Proposition 17.1. *If S is the $(n+4)$-punctured sphere, then $\mathcal{EL}(S)$ satisfies the locally finite n-discs property.*

Our proof is modeled on the proof, due to Andrzej Nagorko [HP], that $\mathbb{R}^{2n+1}_n$ satisfies the locally finite n-discs property. In particular, we modify his notions of *participates* and *attracting grid*.

From now on S will denote the (n+4)-punctured sphere. Let $\mathcal{U}$ be an open cover of $\mathcal{EL}(S)$. Let $\mathcal{U}_2$ denote the refinement of $\mathcal{U}$ produced by Lemma 14.19 and let $\Delta_1, \Delta_2, \cdots$ denote a good cellulation sequence.

Definition 17.2. We say that $\sigma \in \Delta_i$ *participates* in $\mathcal{U}_2$ if $U(\sigma') \in \mathcal{U}_2$ for some face σ' of σ.

Define $A_i = \{\sigma \in \Delta_k, k \leq i | \sigma \text{ participates in } \mathcal{U}_2\}$ and define $\Gamma_i = \mathcal{PML}(S) \setminus \cup_{\sigma \in A_i} \hat{\sigma}$.

Remarks 17.3. i) By Lemma 15.1 and Definition 14.18 each cell of A_i has dimension $\geq n+1$.

ii) Note that Γ_i is obtained from Γ_{i+1} by attaching the cells of $A_{i+1} \setminus A_i$.

Definition 17.4. Call Γ_i the *i'th approximate attracting grid* and $\cap_{i=1}^{\infty} \Gamma_i$ the *attracting grid.*

Definition 17.5. Let $Y_i \subset \mathcal{PML}(S) \setminus \Gamma_i$ denote the *dual cell complex* to A_i. Abstractly it is a simplicial complex with vertices the elements of A_i and $\{v_0, v_1, \cdots, v_k\}$ span a simplex if for all i, $v_i \subset \partial v_{i+1}$.

Remark 17.6. Since $\dim(\mathcal{PML}(S)) = 2n+1$ and each cell of A_i has dimension at least $n+1$ it follows that $\dim(Y_i) \leq n$. This is the crucial fact underpinning the proof of Proposition 17.1.

The next result follows by standard PL topology.

Lemma 17.7. *For every $a > 0$ there exists $N(\Gamma_i)$ a regular neighborhood of Γ_i in $\mathcal{PML}(S)$ such that $N(\Gamma_i) \subset N_{\mathcal{PML}(S)}(\Gamma_i, a)$. Furthermore, there exists a homeomorphism*

$$q : \partial(N(\Gamma_i)) \times [0,1) \to \mathcal{PML}(S) \setminus (\operatorname{int}(N(\Gamma_i)) \cup Y_i)$$

such that $q|\partial N(\Gamma_i) \times 0$ is the canonical embedding and a retraction

$$p : \mathcal{PML}(S) \setminus Y_i \to N(\Gamma_i)$$

that is the identity on $N(\Gamma_i)$, quotients $[0,1)$ fibers to points and has the following additional property. If $\sigma \in A_i$ and $x \in \operatorname{int}(\sigma) \setminus Y_i$, then $p(x) \in \operatorname{int}(\sigma)$. □

Proof of Proposition 17.1. Let $f_i : B^n \to \mathcal{EL}(S), i \in \mathbb{N}$ be a sequence of continuous maps. Being compact $f_i(B^n)$ is covered by finitely many elements $U(\sigma^i_1), \cdots, U(\sigma^i_{i_k})$ of $\mathcal{U}_2$ and hence $\phi^{-1}(f_i(B^n)) \subset \mathcal{PML}(S) \setminus \Gamma_{n_i}$ for some $n_i \in \mathbb{N}$.

Let $\epsilon_i = \min\{\delta(U(\sigma_j)) | \sigma_j \in A_{n_i}\}$ where $\delta(U(\sigma_j))$ is as in Lemma 14.19.

Since both Γ_{n_i} and $\phi^{-1}(f_i(B^n))$ are compact and disjoint it follows that $d_{\mathcal{PML}(S)}(\Gamma_{n_i}, \phi^{-1}(f_i(B^n))) = a_i > 0$.

Claim. *It suffices to show that for each $i \in \mathbb{N}$ there exists $h_i : B^n \to \mathcal{PML}(S)$, a generic PL map, such that for every $t \in B^n$*

i) $d_{\mathcal{PML}(S)}(\Gamma_{n_i}, h_i(t)) < 1/i$ and

ii) $h_i(t) \cup \phi^{-1}(f_i(t)) \subset \hat{\sigma}_j$ for some $\sigma_j \in A_{n_i}$.

Proof. To start with Condition i) implies that for every $z \in \mathcal{EL}(S)$, there exists $\delta_z > 0$ such that $d_{\mathcal{PML}(S)}(h_i(B^n), \phi^{-1}(z)) > \delta_z$ for i sufficiently large. This uses the fact that for each $z \in \mathcal{EL}(S)$, $\phi^{-1}(z)$ is compact and disjoint from Γ_i for i sufficiently large.

Now apply the $\mathcal{EL}$-approximation Lemma 13.3 to the $\{\epsilon_i\}$ and $\{h_i\}$ sequences to obtain maps $g_i : B^n \to \mathcal{EL}(S), i \in \mathbb{N}$ such that

a) For every $z \in \mathcal{EL}(S)$ there exists U_z open in $\mathcal{EL}(S)$ such that $g_i(B^n) \cap U_z = \emptyset$ for i sufficiently large, and

b) if $t \in B^n$, then $d_{PT(S)}(g_i(t), \phi(h_i(t))) < \epsilon_i$.

Conclusion a) gives the local finiteness condition. By ii) of the Claim, if $t \in B^n$, then $h_i(t)$, $\phi^{-1}(f_i(t))$ lie in the same $\hat{\sigma}_j$. Since $\epsilon_i < \delta(U(\sigma_j))$, Lemma 14.19 implies that $g_i(t)$ lies in $U \in \mathcal{U}$ for some $U(\sigma_j) \subset U$. To apply that lemma, let $x = h_i(t)$ and $z = g_i(t)$. Since $f_i(t) \in U(\sigma_j)$ the approximation condition holds too. □

Fix $i > 0$. Let $a_i > 0$ and ϵ_i be as above. Let $a = \min\{1/2i, a_i/2\}$. Using this a, let $p : \mathcal{PML}(S) \setminus Y_{n_i} \to N(\Gamma_{n_i})$ be the retraction given by Lemma 17.7.

By the PML-approximation Lemma 14.11, there exists $h_i' : B^n \to \mathcal{PML}(S)$ such that for each $t \in B^n$, $d_{\mathcal{PML}(S)}(h_i'(t), \phi^{-1}(f_i(t))) \subset \hat{\sigma}$ for some cell σ of Δ_{n_i}. Since $\dim(B^n) + \dim(Y_{n_i}) \le 2n < 2n + 1$ we can assume that $h_i'(B^n) \cap Y_{n_i} = \emptyset$ and this property still holds. Let $h_i = p \circ h_i'$ perturbed slightly to be a generic PL map.

By construction $p(h_i'(B^n)) \subset p(\mathcal{PML}(S) \setminus Y_{n_i}) \subset N(\Gamma_{n_i}) \subset N_{\mathcal{PML}(S)}(\Gamma_{n_i}, a) \subset N_{\mathcal{PML}(S)}(\Gamma_{n_i}, 1/2i)$. Thus Condition i) of the Claim holds for h_i, if it is obtained by a sufficiently small perturbation of $p \circ h_i'$.

By Lemma 17.7 and Lemma 14.10 ii), $p \circ h_i'(t)$ and $\phi^{-1}(z)$ lie in the same $\hat{\sigma}_j$. Thus Condition ii) holds for $p \circ h_i'$. Since each $\hat{\sigma}_j$ is open, this condition holds for any sufficiently small perturbation of $p \circ h_i'$ and so it holds for h_i. This completes the proof of Proposition 17.1 and hence Theorem 16.1. □

After appropriately modifying dimensions, the proof of Proposition 17.1 generalizes to a proof of the following.

Proposition 17.8. *Let $S = S_{g,p}$. Then $\mathcal{EL}(S)$ satisfies the locally finite k-discs property for all $k \le m$, where $m = n - g$, if $p \neq 0$ and $m = n - (g - 1)$, if $p = 0$.* □

18. Applications

By Klarreich [K] (see also [H1]), the Gromov boundary $\partial C(S)$ of the curve complex $C(S)$ is homeomorphic to $\mathcal{EL}(S)$. We therefore obtain the following results.

Theorem 18.1. *Let $C(S)$ be the curve complex of the surface S of genus g and p punctures. Then $\partial C(S)$ is $(n-1)$-connected and $(n-1)$-locally connected. If $g = 0$, then $\partial C(S)$ is homeomorphic to the n-dimensional Nobeling space. Here $n = 3g + p - 4$. Also $\partial \mathcal{C}(S_{2,0}) = \mathbb{R}^5_2$ and $\partial \mathcal{C}(S_{1,2}) = \mathbb{R}^3_1$.*

Remark 18.2. The cases of $S_{0,5}$ and $S_{1,2}$ were first proved in [HP].

Let S be a finite type hyperbolic surface. Let $DD(S)$ denote the space of doubly degenerate marked hyperbolic structures on $S \times \mathbb{R}$. These are the complete hyperbolic structures with limit set all of S^2_∞ whose parabolic locus corresponds to the cusps of S. It is topologized with the algebraic topology. See §6 [LS] for more details. As a consequence of many major results in hyperbolic 3-manifold geometry, Leininger and Schleimer proved the following.

Theorem 18.3. [LS] *$DD(S)$ is homeomorphic to $\mathcal{EL}(S) \times \mathcal{EL}(S) \setminus \Delta$, where Δ is the diagonal.*

Corollary 18.4. *If S is the $(n+p)$-punctured sphere, then $DD(S)$ is homeomorphic to $\mathbb{R}^{2n+1}_n \times \mathbb{R}^{2n+1}_n \setminus \Delta$. In particular $DD(S)$ is $(n-1)$-connected and $(n-1)$-locally connected.*

The subspace of marked hyperbolic structures on $S \times \mathbb{R}$ which have the geometrically finite structure Y on the $-\infty$ relative end is known as the Bers slice B_Y. As in [LS], let $\partial_0 B_Y(S)$ denote the subspace of the Bers slice whose hyperbolic structures on the ∞ relative end are degenerate.

Theorem 18.5. [LS] *$\partial_0 B_Y(S)$ is homeomorphic to $\mathcal{EL}(S)$.*

Corollary 18.6. *If S is a hyperbolic surface of genus-g and p-punctures, then $\partial_0 B_Y(S)$ is $(n-1)$-connected and $(n-1)$-locally connected. If $g = 0$, then $\partial_0 B_Y(S)$ is homeomorphic to the n-dimensional Nobeling space. Here $n = 3g + p - 4$.*

Theorem 18.7. *If S is a p-punctured sphere, $p \geq 5$, $S_{2,0}$ or $S_{1,2}$, then there exists a simple closed curve α in $\mathcal{EL}(S)$ such that $\phi^{-1}(\alpha)$ contains no simple closed curve that projects to α under ϕ.*

Proof. By the proof of Theorem 9.1 [G1] there exists a 1-simplex $\sigma \subset \mathcal{PML}(S)$ such that $\phi(\sigma) = z \in \mathcal{EL}(S)$ and $\sigma = \lim \eta_i$ where η_i is a 1-simplex in $\mathcal{PML}(S)$ consisting of those projective measured laminations supported on two disjoint simple closed geodesics. By Theorem 9.1 [G1] each η_i is a limit of 1-simplices κ_i with $\phi(\kappa_i) \in \mathcal{EL}(S)$. Thus $\sigma = \lim \sigma_i$ where $\phi(\sigma_i) = z_i \in \mathcal{EL}(S)$ and $z, z_1, z_2, \cdots$ are distinct. Let $\partial\sigma = x \cup y$ and $\partial\sigma_i = x_i \cup y_i$. We can assume that $x_i \to x$ and $y_i \to y$. Let β_i (resp. γ_i) be a generic PL path from x_i to x_{i+1} (resp. y_i to y_{i+1}). We can assume that $\lim(\beta_i) = x$, $\lim(\gamma_i) = y$ and using [Ma], [Ve] or [Ke] that for each $i \in \mathbb{N}$ there exist $p_i \in \beta_i$ and $q_i \in \gamma_i$ such that $\phi(p_i)$ and $\phi(q_i)$ are uniquely ergodic. Furthermore, the various $\phi(p_i)$'s and $\phi(q_j)$'s are distinct. The σ_i's can be oriented so that the concatenation $\alpha_1 * \sigma_2 * \beta_2 * \sigma_3 * \alpha_3 * \cdots$ is a well defined (topologist sine curve like) path defined on $[-2, 0)$ that limits on σ. Appropriately applying Lemma 13.3 we obtain a path $g : [-2, 0] \to \mathcal{EL}(S)$ such that $g(-2) = z_1$ and for $i \in \mathbb{N}$, $g(-1/(2i-1)) = \phi(p_{2i-1})$ and $g(-1/(2i)) = \phi(q_{2i})$ and $g(0) = z$. Since $\mathcal{EL}(S)$ is a Nobeling space, we readily find an *embedded* path $f : [-2, 0] \to \mathcal{EL}(S)$ with the same properties that additionally is the concatenation of countably many subpaths limiting to z such that on each subpath all but one coordinate of $\mathbb{R}^{2n+1}$ is constant. Observe that f extends to an embedding $F : S^1 \to \mathcal{EL}(S)$, where S^1 is the natural quotient of $[-2, 2]$. By construction, there exists no continuous map $h : S^1 \to \mathcal{PML}(S)$ such that $\phi(h(t)) = F(t)$, thus $\phi^{-1}(F(S^1))$ contains no simple closed curve that projects to $F(S^1)$. $\square$

19. Problems and Conjectures

To start with we restate the following long-standing classical question.

Problem 19.1. *Find a topological characterization of the Nobeling type space $\mathbb{R}^p_q$, the space of points in $\mathbb{R}^p$ with at most q rational coordinates.*

Remarks 19.2. i) The fundamental work of Ageev [Av], Levin [Le] and Nagorko [Na] solve this problem for $p = 2m + 1$ and $q = m$.

ii) Of particular interest are the spaces of the form $\mathbb{R}^{2n+1}_m$, where $m \geq n$.

iii) In analogy to Theorem 16.5, does it suffice that X be separable, complete metric, q-dimensional, $(q-1)$-connected, $(q-1)$-locally connected, and satisfy the locally finite $(p-q-1)$-discs property?

We offer the following very speculative;

Conjecture 19.3. *Let S be a p-punctured surface of genus-g. Then $\mathcal{EL}(S)$ is homeomorphic to $\mathbb{R}^{2n+1}_{n+k}$, where $n = 3g+p-4$ and $k = 0$ if $g = 0$, $k = g-1$ if both $p \neq 0$ and $g \neq 0$, and $k = g-2$ if $p = 0$.*

The value of k in this conjecture is motivated by the following duality conjecture.

Conjecture 19.4. *(Duality Conjecture) If S is the p-punctured surface of genus-g, then*

1) $\dim(EL) + h\dim(\mathcal{C}(S)) + 1 = \dim(\mathcal{PML}(S))$, *where* $h(\dim(\mathcal{C}(S)))$ *is the homological dimension of the curve complex.*

If $\dim(\mathcal{PML}(S)) = 2n+1$, *then*

2) for every m, there is a natural injection $H^{st}_m(\mathcal{EL}(S)) \to H^{2n-m}(\mathcal{C}(S))$ and

3) for every m, there is a natural injection of $H_m(\mathcal{C}(S)) \to \check{H}^{2n-m}(\mathcal{EL}(S))$.

Remark 19.5. Harer [Hr] computed the homological dimension of $\mathcal{C}(S)$, hence duality conjecture 1) is equivalent to the conjecture that $\dim(\mathcal{EL}(S)) = n$ if $g = 0$; $\dim(\mathcal{EL}(S)) = n + (g-1)$ if $p \neq 0$ and $g \neq 0$; and $\dim(\mathcal{EL}(S)) = n + (g-2)$ if $p = 0$.

Remarks 19.6. In the above conjectures, homology is Steenrod and cohomology is Cech. Both coincide with the corresponding singular theory for CW-complexes. When $m = 0$ or $2n - m = 0$, the above is reduced homology or cohomology.

Except for a single dimension all the maps and groups should be trivial.

The conjecture is motivated by the observation that $\mathcal{EL}(S)$ and $\mathcal{C}(S)$ *almost live* in $\mathcal{PML}(S) = S^{2n+1}$ as complementary objects and hence Alexander - Sitnikov duality [Sit] applies. Indeed, since $\mathcal{PML}(S)$ is the disjoint union of $\mathcal{FPML}(S)$ and $\mathcal{UPML}(S)$ the following holds by Alexander - Sitnikov duality.

Theorem 19.7. *If S is a finite type surface and* $\dim(\mathcal{PML}(S)) = 2n+1$, *then*

1) $H^{st}_m(\mathcal{FPML}(S))$ is isomorphic to $\check{H}^{2n-m}(\mathcal{UPML}(S))$ and

2) $H^{st}_m(\mathcal{UPML}(S))$ is isomorphic to $\check{H}^{2n-m}(\mathcal{FPML}(S))$. □

Remarks 19.8. Again we use reduced homology and cohomology for $m = 0$ or $2n - m = 0$.

Now $\mathcal{UPML}(S)$ has a CW-structure $\mathcal{UPML}(S)_{cw}$ and there is an inclusion $\mathcal{C}(S) \to \mathcal{UPML}(S)_{\mathrm{cw}}$ which induces a homotopy equivalence [G2]. However, the topology on $\mathcal{UPML}(S)_{\mathrm{cw}}$ is finer than that of $\mathcal{UPML}(S)$ for essentially the same reason that the topology on $\mathcal{C}(S)$ is finer than that of $\mathcal{C}(S)_{sub}$. See Remark 1.21. There is the analogy that $\mathcal{C}(S)$ is to $\mathcal{C}(S)_{sub}$ as the infinite wedge of circles is to the Hawaiian earings. Note that the inclusion of the former into the latter induces proper injections of H_1 and H^1. For that reason the maps in Conjecture 19.4 are injections rather than isomorphisms.

The map $\phi : \mathcal{FPML}(S) \to \mathcal{EL}(S)$ is a closed map (Corollary 1.14) and point inverses are cells (Lemma 1.17); hence by [Sp] or [Sk] we obtain the following result.

Theorem 19.9. *Let S be a finite type surface. Then for all m*

1) $\phi_* : H^{st}_m(\mathcal{FPML}(S)) \to H^{st}_m(\mathcal{EL}(S))$ is an isomorphism and

2) $\phi^* : \check{H}^m(\mathcal{EL}(S)) \to \check{H}^m(\mathcal{FPML}(S))$ is an isomorphism.

Corollary 19.10. *Let S be a finite type surface and $\dim(\mathcal{PML}(S)) = 2n+1$, then for all m there are natural isomorphisms*

1) $\check{H}^m(\mathcal{EL}(S)) \cong H^{st}_{2n-m}(\mathcal{UPML}(S))$ and

2) $H^{st}_m(\mathcal{EL}(S)) \cong \check{H}^{2n-m}(\mathcal{UPML}(S))$.

Remark 19.11. It follows from Corollary 19.10 and Remarks 19.8 that to prove parts 2) and 3) of the duality conjecture it suffices to show that the bijective map $I : \mathcal{UPML}(S)_{\mathrm{cw}} \to \mathcal{UPML}(S)$ induces inclusions in Steenrod homology and Cech cohomology. That will also be a big step towards proving 1), since it implies that $\dim(\mathcal{EL}(S))$ is at least as large as the conjectured value.

Conjecture 19.12. *Let S be a finite type surface. The bijective maps $I : \mathcal{C}(S) \to \mathcal{C}(S)_{\mathrm{sub}}$ and $I : \mathcal{UPML}(S)_{\mathrm{cw}} \to \mathcal{UPML}(S)$ induce inclusions in Steenrod homology and Cech cohomology.*

Question 19.13. *Let S be a finite type surface. Is $\mathcal{EL}(S)$ m-connected (resp. m-locally connected) if and only if $\mathcal{FPML}(S)$ is m-connected (resp. m-locally connected).*

Remark 19.14. It follows from Theorem 1.18 that $\mathcal{EL}(S)$ is connected if and only if $\mathcal{FPML}(S)$ is connected.

Definition 19.15. Let X be a regular cell complex. We say that X is *super symmetric* if $X/\operatorname{Isom}(X)$ is a finite complex.

Example 19.16. Examples of such spaces are the curve complex and arc complex of a finite type surface and the disc complex of a handlebody.

Question 19.17. *What spaces arise as the boundaries of super symmetric Gromov-hyperbolic regular cell complexes? Under what conditions does the m-dimensional Nobeling space arise? More generally, under what conditions does a Nobeling type space arise.*

Remark 19.18. This is a well known question, when the cell complex is a Caley graph of a finitely generated group. Indeed, Kapovich and Kleiner give an essentially complete answer when ∂G is 1-dimensional [KK]. The Gromov boundaries of locally compact Gromov hyperbolic spaces are locally compact, thus the point of this question is to bring attention to non locally finite complexes. Note that Gromov and Champetier [Ch] assert that the *generic* finitely presented Gromov-hyperbolic group has the Menger curve as its boundary.

Question 19.19. *Let $x \in \mathcal{EL}(S)$ be non uniquely ergodic. Does $\phi^{-1}(x)$ arise as in the construction of Theorem 9.1* [G2] *or as limits of cells of the form $B_{a_1} \cap \cdots \cap B_{a_k}$ where $a_1, \cdots, a_k$ are pairwise disjoint simple closed curves?*

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