COMMENTARY ON FOLIATIONS*

Quoting Thurston's definition of *foliation* [F11]. "Given a large supply of some sort of fabric, what kinds of manifolds can be made from it, in a way that the patterns match up along the seams? This is a very general question, which has been studied by diverse means in differential topology and differential geometry. ... A foliation is a manifold made out of striped fabric - with infintely thin stripes, having no space between them. The complete stripes, or *leaves*, of the foliation are submanifolds; if the leaves have codimension k, the foliation is called a codimension k foliation. In order that a manifold admit a codimension-k foliation, it must have a plane field of dimension $(n-k)$." Such a foliation is called an $(n-k)$-*dimensional foliation*.

The first definitive result in the subject, the so called *Frobenius integrability theorem* [Fr], concerns a necessary and sufficient condition for a plane field to be the tangent field of a foliation. See [Spi] Chapter 6 for a modern treatment. As Frobenius himself notes [Sa], a first proof was given by Deahna [De]. While this work was published in 1840, it took another hundred years before a geometric/topological theory of foliations was introduced. This was pioneered by Ehresmann and Reeb in a series of *Comptes Rendus* papers starting with [ER] that was quickly followed by Reeb's foundational 1948 thesis [Re1]. See Haefliger [Ha4] for a detailed account of developments in this period.

Reeb [Re1] himself notes that the 1-dimensional theory had already undergone considerable development through the work of Poincare [P], Bendixson [Be], Kaplan [Ka] and others. In addition there was the well known extension to the Poincare - Hopf index theorem: a closed manifold has Euler characteristic 0 if and only if it has a nowhere vanishing smooth vector field if and only if it has a 1-dimensional foliation. Another impetus was Hopf's question as to whether or not the 3-sphere has a codimension-1 foliation [Re2].

The foliation exhibited in Reeb's thesis, now known as the *Reeb foliation* gave a positive answer to Hopf's question. At the outset Reeb asks the following fundamental and far-reaching generalization to Hopf's question [Re1]: *Si la variete V_n admet un champ E_q de classe C^1 admet-t-elle aussi un champ E^q de classe C^1 completement integrale?* which experts soon after expressed as follows: Is every q-plane field homotopic to the tangent plane field of a foliation?

In a series of three papers [F7], [F10], [F12] Thurston obtained the following results.

Theorem 0.1. [F12] [F10] *Let M be a smooth manifold without boundary. Every codimension-1 plane field on M is homotopic to the tangent plane field of a C^∞ codimension-1 foliation.*

Corollary 0.2. [F12] *Every closed, connected. smooth manifold with Euler characteristic 0 has a C^∞ codimension-1 foliation.*

Theorem 0.3. [F7] *Every smooth k-plane field on a closed n-manifold is homotopic to the tangent plane field of a Lipschitz foliation, with C^∞-leaves.*

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The Lipschitz condition was improved to C^1 by Tsuboi [Ts1]. Thurston uses the Bott vanishing theorem [Bo] in [F5] to show that there cannot be a C^2-version of this theorem and further that the dimension obstruction given by Bott is sharp. See [Mo] for an explicit example. For 2-plane fields we have the following result.

Theorem 0.4. [F7] *Every C^∞ 2-plane field on a manifold is homotopic to a completely integrable C^∞ plane field.*

Remark 0.5. In [F12], after mentioning various results of Lickorish, Novikov-Zieschang, Wood, Lawson, Durfee, A'Campo and Tamura, Thurston states, "My method, on the other hand, is local in nature: one cannot see the whole manifold. A disadvantage is that it is hard to picture the foliations so constructed. For this reason, I think that further work on the geometric methods of constructing foliations is called for."

The proofs of these results required generalizations of deep results of Mather [Ma3], [Ma1] on the group of compactly supported diffeormorphisms of $\mathbb{R}$ with the discrete topology and relations with Haefliger's classifying spaces. In particular Thurston proved the following.

Theorem 0.6. [F5] *If M is a closed manifold, then $\mathrm{Diff}_0^\infty(M)$ is a simple group.*

Theorem 0.7. [F5] *For all $r \geq 1$ and $p \geq 1$, $r, p \in \mathbb{N}$, there is a map*

$$\bar{B}\,\mathrm{Diff}_c^r(\mathbb{R}^p) \to \Omega^p(B\bar{\Gamma}_p^r)$$

that induces an isomorphism on integer homology.

Here Diff_0^∞ denotes the connected component of the identity of the group of C^∞ diffeomorphisms, Diff_c^r denotes compacted supported C^r diffeomorphisms and $B\bar{\Gamma}_p^r$ is defined in Remark 0.13. Note that $\bar{B}\,\mathrm{Diff}_c^r(\mathbb{R}^p)$ is also a homotopy fiber. Theorem 0.7 is due to John Mather for $p = 1$. The paper [F5] is a research announcement with few hints of proofs. Thurston lectured on this at Harvard in 1974 and Mather wrote a proof of the above theorem in [Ma2]. See also [Sar], [McD], [La], and [Ts2]. One of Thurston's proofs introduced the far reaching technique now known as *fragmentation*, see [Na].

Forty years later Gael Meigniez [Me] proved Theorem 0.1 for $n \geq 4$ without using the Mather - Thurston theory. In addition his foliations are *minimal*, i.e. every leaf is dense. This includes simply-connected manifolds and connect sums, in contrast to Novikov's closed leaf theorem [No], which disallows minimal foliations in such 3-manifolds.

There are also relative versions of Thurston's theorems, e.g. given a manifold M with boundary, then under suitable circumstances a plane field defined on M that is tangent to a foliation near ∂M, is homotopic rel a neighborhood of ∂M to a completely integral plane field. Reeb's stability theorem precludes this holding in complete generality, for if ∂M is simply connected, then any foliation on M with ∂M a leaf would be foliated by leaves covered by ∂M. However, if each component N of ∂M satisfies $H^1(N, \mathbb{R}) \neq 0$, then extension theorems hold. Consult [F7], [F12] for exact statements. Thurston proved the following generalization of the Reeb stability theorem.

Theorem 0.8. [F6] *(a) Let $\mathcal{F}$ be a codimension-1 C^1 transversely oriented foliation on a compact manifold M whose (possibly empty) boundary is a union of leaves. Suppose L^{n-1} is a compact leaf of $\mathcal{F}$ such that $H^1(L, \mathbb{R}) = 0$. Then every leaf of $\mathcal{F}$ is diffeomorphic with L^{n-1} and M fibers over S^1 or $[0, 1]$ with fiber L^{n-1}.*

(b) Let $\mathcal{F}$ be a C^1 codimension-k foliation. If L is a compact leaf of $\mathcal{F}$ with trivial linear holonomy and $H^1(L,R)=0$, then L has trivial holonomy and hence L has a tubular neighborhood which fibers over D^k with leaves as fibers.

Remark 0.9. Using hyperbolic geometry Thurston demonstrated a C^0 counterexample to (a), nevertheless he states, "It would be interesting to have a characterization of compact leaves L for which Reeb's [stability] conclusion holds in the C^0 case."

Remark 0.10. Haefliger [Ha1] showed that the 3-sphere does not support an analytic codimension-1 foliation, thus Theorems 0.1, 0.2, and 0.4 are not applicable to analytic foliations. Thurston notes [F12] that "His [Haefliger's] class of counterexamples was somewhat enlarged by Novikov [No] and Thurston [F1], but the theory of analytic foliations still has many unanswered questions."

We now present background to Thurston's work on Haefliger structures and homotoping smooth plane fields to smooth foliations. Let M by an n-dimensional manifold and $\mathcal{F}$ a C^r codimension-q foliation. Let ν denote the normal bundle to $\mathcal{F}$, an $\mathbb{R}^q$-bundle over M. The exponential map gives a submersion from a neighborhood of the 0-section in ν to M. Pulling back $\mathcal{F}$ gives a codimension-q foliation $\mathcal{G}$ of ν transverse to the fibers. As the zero section $s: M \to \nu$ is transverse to $\mathcal{G}$ we can recover $\mathcal{F}$ as the intersection of $s(M)$ with $\mathcal{G}$.

A *Haefliger structure* [Ha2], [Ha3], [Ha4] is a $\mathbb{R}^q$-bundle E over M together with a section $s: M \to E$ and a germ near $s(M)$ of a C^r codimension-q foliation $\mathcal{G}$ of E transverse to the fibers of E. Two Haefliger structures are *concordant* if they cobound a Haefliger structure on $M \times [0,1]$. Unlike foliations, Haefliger structures pull back under continuous maps and may be studied with the tools of algebraic topology. Indeed, Haefliger showed that C^r codimension-q Haefliger structures on M, up to concordance, correspond bijectively with homotopy classes of maps from M to the classifying space $B\Gamma_q^r$, where Γ_q^r is the groupoid of germs of C^r diffeomorphisms of $\mathbb{R}^q$ with a certain topology. This work was motivated by Bott's vanishing theorem and work of [Ph1], [Ph2], that grew out of the Smale - Hirsch immersion theory. Using the work of Gromov [Gr] and Phillips [Ph3], Haefliger further showed [Ha3] that if M is *open*, then every Haefliger structure is concordant to one arising from a foliation. One of Thurston's major accomplishments was to prove similar results for *closed* manifolds and certain relative cases. Recall that codimension-k foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ are *concordant* if there exists a codimension-k foliation on $M \times I$ which restricts to $\mathcal{F}_0$, $\mathcal{F}_1$ respectively on $M \times 0$ and $M \times 1$.

Theorem 0.11. [F7], [F12] *Concordance classes of foliations on the closed manifold M correspond 1-1 with homotopy classes of Haefliger structures $\mathcal{H}$ together with concordance classes of bundle monomorphisms $i: v_{\mathcal{H}} \to T(M)$.*

Theorem 0.12. [F7], [F12] *(see also P. 347* [Ha5]*) If K is a smooth p-plane field of codimension-$q \geq 1$ on the closed manifold M, then K is homotopic to a smooth completely integrable p-plane field if and only if the map of M into BGl_q classifying the normal bundle to K can be lifted to a map into $B\Gamma_q^r$, where $D: B\Gamma_q^r \to BGl_q$ is induced by the differential.*

See [F7] and [F12] for more results in this direction and other interesting applications. These last two theorems were subsequently proved for codimension-$q \geq 2$ using *wrinkled mappings* by Eliashberg and Mishachev with extensions to families of foliations [EM1], [EM2].

Remark 0.13. Haefliger remarks, P.348 [Ha5], that despite the spectacular and profound results of the 70's, much remains to be understood about the homotopy type of the spaces $B\bar{\Gamma}_q^r$, the homotopy fibers of D.

Question 0.14. *(Haefliger* [Ha5]*)*
a) Is $B\bar{\Gamma}_q^r$ connected for $q > 1$?
b) What is the first p for which $\pi_p(B\bar{\Gamma}_q^r) \neq 1$?

Conjecture 0.15. *(Thurston* [F5]*) The smallest k for which $H_{k+p}(B\bar{\Gamma}_p^\infty; \mathbb{Z}) \neq 0$ is $p + 1$.*

Remark 0.16. [F5] When $p = 1$, then $k > 1$ by Mather [Ma3] and $k \leq p + 1$ using a generalized Godbillon - Vey invariant.

A central question is to understand how to distinguish different classes of foliations on a given manifold M. The Godbillon - Vey invariant [GV] provides an invariant for smooth foliations on 3-manifolds, defined as follows. If $T(\mathcal{F})$ denotes the tangent plane field of $\mathcal{F}$, then it is the kernal of a 1-form α. The Frobenius theorem implies that $\alpha \wedge d\alpha = 0$ so that $d\alpha = \alpha \wedge \theta$ for some θ. The *Godbillion - Vey* form is the closed 3-form $\theta \wedge d\theta$. It is a concordance invariant of foliations, indeed of Haefliger structures.

In [F3] Thurston cryptically states "The form $\theta \wedge d\theta$ [the Godbillon - Vey class] may be interpreted as a measure of the helical wobble of the leaves of $\mathcal{F}$..." . An interpretation, attributed to him is as follows. The form θ can be viewed as the logarithmic derivative of the rate at which leaves spread apart under the holonomy; it is dual to a vector field to $\mathcal{F}$ in the direction of maximal contraction. As one moves transverse to the foliation the Godbillion - Vey form measures infinitesimally the algebraic area of the swept by the vectors under this motion. See [Pi] for more details. Thurston proves the following result using the geodesic flow on the unit tangent bundle of hyperbolic 2-space.

Theorem 0.17. [F3] *There are uncountably many noncobordant codimension-1 C^∞-foliations of S^3. The Godbillon-Vey invariant induces a surjective homomorphism of $\pi_3(B\Gamma_1^r)$ onto $\mathbb{R}$ for $2 \leq r \leq \infty$.*

In contrast to Theorem 0.4 there is Thurston's unpublished 1971 Ph. D. thesis [F1].

Theorem 0.18. [F1] *Let M^3 be a circle bundle over a closed surface. Any C^2-foliation of M either has a compact leaf or can be isotoped to be transverse to the fibers.*

As a corollary, using the Milnor - Wood inequality, it follows that if M is a circle bundle over a surface S and the Euler class of the bundle is $> |\chi(S)|$, then any C^2-foliation of M has a compact leaf, thereby giving the first closed aspherical 3-manifolds with that property. Thurston shows that both the theorem and corollary are false for C^0-foliations. Here is how he constructs a foliation without compact leaves on any circle bundle M over a surface S of genus ≥ 2. Start with a transversely orientable geodesic lamination $\lambda \subset S$ without compact leaves and only 4-prong disc complementary regions. Extend to a foliation of M by first suspending λ in the S^1 direction and then filling in the complementary regions with bundles of saddles. Note that geodesic laminations play a central role in his future work on homeomorphisms of surfaces as well as geometrization. The thesis also gives a counterexample to an assertion of Novikov that a certain partial order within Novikov components of a foliation takes on a minimum.

An important technical result in [F1], proved independently by Roussarie [Ro], is that an embedded incompressible surface in a C^2-taut foliation can be isotoped so that each component is either a leaf or transverse to the foliation except for finitely many saddle tangencies. By *taut* we mean a transversely orientable foliation such that every leaf intersects a closed transversal. Roussarie and Thurston proved a version for manifolds with boundary and Roussarie a version for transversely orientable foliations without generalized Reeb components. Implicit in [Ro] and stated in [F14] is the generalization to *Reebless foliations*, i.e. foliations having no Reeb components. Here, the isotoped surface may also have finitely many circle tangencies. Using [Ca2], this technical result and its proof extend to C^0-foliations. It also holds for immersed π_1-injective surfaces [Ga2].

More distinctions between C^0 and C^2 foliations are given in the paper [F4] with H. Rosenberg. Here is one.

Theorem 0.19. [F4] *There exist smooth foliations $\mathcal{F}_0, \mathcal{F}_2$ on a closed 3-manifold such that $\mathcal{F}_0$, $\mathcal{F}_2$ are C^0-concordant but not C^2-cobordant.*

Thurston's two papers with Plante are about foliations and growth rates of groups. The paper [F2] shows that the fundamental group of a compact Riemannian manifold with a codimension-1 Anosov flow has exponential growth. Paper [F13] shows that if the holonomy of a leaf of a codimension-1 foliation has polynomial growth, then it is virtually nilpotent. While the main result is true for all groups by the celebrated result of Gromov [Gr2], the paper is full of interesting observations. For example,

Theorem 0.20. [F13] *If G is a group of C^2-diffeomorphisms of $[0,\infty)$ of polynomial growth, then G is free abelian.*

Thurston's (unique) paper with his thesis advisor Morris Hirsch [F8] was motivated by a paper by Plante [Pl] that in turn was a generalization of the Poincare - Bendixson theorem. As an application of their main result they prove:

Theorem 0.21. [F8] *If M is a compact flat manifold whose fundamental group is obtained by taking free products and finite extensions of solvable groups, then $\chi(M) = 0$.*

This implies a special case of the so called *Chern conjecture* which asserts that $\chi(M) = 0$ for all closed affine manifolds.

The unpublished 1997 ArXiv preprint, *Three-manifolds, Foliations and Circles, I* [F15] introduced the notion of a manifold *slithering* over another and in particular that of a 3-manifold M slithering over the circle. In that case the universal cover $\tilde{M}$ of M fibers over S^1 so that the deck transformations are bundle automorphisms and the fibers are unions of leaves of $\tilde{\mathcal{F}}$, where $\mathcal{F}$ is a taut foliation of M. Thurston shows that $\tilde{\mathcal{F}}$ has space of leaves $\mathbb{R}$, i.e. $\mathcal{F}$ is $\mathbb{R}$-*covered*, and any two leaves of $\tilde{\mathcal{F}}$ are at bounded Hausdorff distance, i.e. $\tilde{\mathcal{F}}$ is *uniform*. Furthermore he shows that if $\tilde{\mathcal{F}}$ has this property then M slithers over S^1. He conjectured that $\mathbb{R}$-covered foliations are uniform, but a counterexample was found in Calegari [Ca1]. Thurston states that non Haken 3-manifolds that have such slitherings, arising from Fenley's $\mathbb{R}$-covered Anosov foliations [Fe]. He shows that a slithered foliation $\mathcal{F}$ possesses much structure analogous to that of a fibration. For example, M supports a genuine lamination transverse to $\mathcal{F}$ and the ends of the leaves of $\tilde{\mathcal{F}}$ organize to a single circle on which $\pi_1(M)$ acts.

The abstract of [F15] and the unfinished paper [F16] assert that the first property and an analogue of the second holds for any taut foliation $\mathcal{F}$ on any closed atoroidal 3-manifold M. In particular, Thurston proves the following.

Theorem 0.22. *(Thurston)* [CD] *Let $\mathcal{F}$ be a taut foliation of an orientable 3-manifold M with hyperbolic leaves with $\tilde{\mathcal{F}}$ denoting its preimage in $\tilde{M}$. Then there exists a universal circle S^1_{univ} for $\mathcal{F}$. I.e. there exists an action of $\pi_1(M)$ on S^1_{univ} with the property that for each leaf L of $\tilde{\mathcal{F}}$, there exists a quotient map $p_L : S^1_{univ} \to S^1_\infty(L)$ natural with respect to the action of $\pi_1(M)$ on S^1_{univ}. (See 6.1* [CD] *for the precise definition of S^1_{univ}.)*

Theorem 0.23. *(Calegari - Dunfield* [CD]*) If M is atoroidal, then the action of $\pi_1(M)$ on S^1_{univ} is faithful.*

In Spring 1997, Thurston gave several three-hour lectures on the construction of the universal circle at the *Very Informal Foliations Seminar* at the MSRI and most of [F16] is concerned with a construction on S^1_{univ}. Based on these lectures, Danny Calegari and Nathan Dunfield wrote a careful account of Theorem 0.22, see §5 - §6 of [CD]. The key technical lemma, Thurston's *leaf pocket theorem* proven in §5 roughly asserts that for any leaf L of $\mathcal{F}$, the holonomy is defined along most rays in L. (By Candel [Can], there exists a Riemannian metric on M for which each leaf has constant negative curvature, hence $\partial\tilde{L}$ is naturally identified with a circle.) The proof of the leaf pocket theorem in [CD] is topological as opposed to Thurston's original proposed proof using various properties of Lucy Garnett's harmonic measures [Gar] and is extended to essential laminations. It is used to show that the fundamental group of many manifolds including the Weeks manifold do not act faithfully on the circle, hence do not support a taut foliation. Earlier, Roberts, Shareshian and Stein [RSS] found examples of closed hyperbolic 3-manifolds that do not support taut foliations.

Inspired by [F15], D. Calegari proved the following.

Theorem 0.24. *(Calegari* [Ca3]*) A taut foliation of a closed, orientable,algebraically atoroidal 3-manifold is either the weak-stable foliation of an Anosov flow, or else there are a pair of very full genuine laminations transverse to the foliation.*

Thurston's *A norm for the homology of 3-manifolds* was a 1976 preprint published 10 years later. For a 3-manifold M with (possibly empty) boundary, this seminal paper introduced a norm on $H_2(M, \partial M; \mathbb{R})$ that generalizes the notion of genus of a knot. It is non-degenerate if M is atoroidal. The unit ball is a finite sided polyhedron and the set of homology classes realized by fibers of a fibration over S^1 correspond to a finite (possibly empty) set of open top dimensional faces in that all the lattice points lying in rays through these faces are representable by fibers and conversely. Thurston further proves that leaves of taut (and more generally Reebless) foliations are homologically norm minimizing. In contrast, about a year later Sullivan [Su] proved that leaves of a C^2 taut foliation $\mathcal{F}$ are homologically area minimizing. In fact $\mathcal{F}$ can be calibrated [HL]. *Homologically area minimizing* means that if L is a leaf of $\mathcal{F}$ and $R \subset L$ is a smooth compact subsurface, then area$(R) \leq$ area(S) for all compact surfaces S with $\partial S = \partial R$ and $[S] = [R]$ rel ∂R. While stated for compact leaves, the analogous subsurface property also holds for homological tautness.

Theorem 0.25. [F14] *Let M be a compact orientable 3-manifold with a transversely orientable Reebless foliation $\mathcal{F}$ transverse to ∂M. If L is a compact leaf of $\mathcal{F}$, then L is norm minimizing in its homology class as an element of $H_2(M, \partial M)$.*

Theorem 0.26. *(Sullivan)* [Su] *If $\mathcal{F}$ is a C^2-taut foliation of the 3-manifold M, then there exists a Riemannian metric such that every leaf is homologically area minimizing.*

Thurston proved Theorem 0.25 by observing that the Euler class $\chi(\mathcal{F})$ of a foliation $\mathcal{F}$ evaluated on a compact leaf L gives up to sign $|\chi(L)|$. On the other hand, if $\mathcal{F}$ is taut and T is an incompressible surface, then using Rousserie - Thurston general position [Ro], [F1], T can be isotoped to be transverse to $\mathcal{F}$ except at saddle tangencies. Thus if T is homologous to L, evaluating the Euler class on T via obstruction theory shows that $|\chi(L)| \leq |\chi(T)|$. Thurston conjectured a similar Euler class type inequality for contact structures (compare Theorem 3.8 [ET1]) that was proved by Bennequin [Ben].

If k is a knot in $\mathbb{R}^3$ transverse to the contact structure τ and F is a Seifert surface for k, then $\tau|F$ is trivial. Pushing k off itself using a non vanishing section of that bundle gives a knot k'. Define the *self-linking number* $sl(k)$ to be the linking number of k' with k.

Theorem 0.27. *(Bennequin) Let τ denote the standard contact structure on $\mathbb{R}^3$ and k a knot transverse to τ. If F is an oriented Seifert surface for k, then $sl(k) \leq -\chi(S)$.*

Bennequin used this result to show that there exist non-standard contact structures on $\mathbb{R}^3$, giving the first example of what are now called *overtwisted* contact structures. Eliashberg introduced the dichotomy of *tight* and overtwisted contact structures [El] and proved Theorem 0.27 for null homologous knots transverse to tight contact structures in general 3-manifolds as well as a version for null homologous Legendrian knots in tight contact structures.

To bridge the theory of foliations with contact structures in oriented 3-manifolds, Thurston and Eliashberg [ET2] introduced the theory of *positive (resp. negative) confoliations*, i.e. those having plane fields annihilated by 1-forms α, such that $\alpha \wedge d\alpha \geq 0$ (resp. $\alpha \wedge d\alpha \leq 0$). They proved the following foundational result.

Theorem 0.28. *Any C^2-confoliation ψ on an oriented 3-manifold $\neq S^2 \times S^1$ can be C^0-approximated by contact structures. If ψ is a foliation, then it can be approximated by both positive and negative contact structures. Contact structures C^0-close to C^2-taut foliations are symplectically fillable and hence tight.*

This theorem for C^0 taut foliations was independently proved by Bowden [Bo] and Kazez-Roberts [KR].

The following converse to Theorem 0.25 was proven by Gabai.

Theorem 0.29. [Ga2] *If M is a closed, orientable, irreducible, atoroidal 3-manifold and S is a Thurston norm minimizing surface, then there exists a C^∞ taut foliation on M having S as a leaf.*

See [Ga2], [Ga3] for finite depth versions as well as ones for manifolds with boundary. Thurston's conjecture [F14] that the norm based on embedded surfaces is equal to the norm based on singular (i.e. mapped) surfaces was proven in [Ga1]. This led to a proof that the Gromov norm on $H_2(M, \partial M, \mathbb{R})$ is equal to twice the Thurston norm as well as a generalization of the loop and sphere theorems to higher genus surfaces. Efforts to prove these theorems led to Gabai's theory of sutured manifold hierarchies [Ga2], [Ga1] and a proof of a strong form of the Property R conjecture, that for a knot in the 3-sphere a minimal genus surface extends to a Thurston norm minimizing surface under 0-frame surgery. A counterexample to Conjecture 3 of [F14] is given in [Ya] and [GY], though it follows from [Ga2] that there is a positive solution for vertices of the unit ball of the dual Thurston norm. See Theorem 1.4 [GY].

The Thurston norm and its relation to fibrations and more generally taut foliations has become fundamental to knot theory, low dimensional topology, theory of contact structures, foliation and lamination theory, geometric group theory, symplectic topology, dynamical systems, as well as gauge theory and Heegaard Floer homology and their variants.

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