Interpolation and Extrapolation of Smooth Functions by Linear Operators

by

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Abstract: Let $C^{m,1}(\mathbb{R}^n)$ be the space of functions on $\mathbb{R}^n$ whose m^{th} derivatives are Lipschitz 1. For $E \subset \mathbb{R}^n$, let $C^{m,1}(E)$ be the space of all restrictions to E of functions in $C^{m,1}(\mathbb{R}^n)$. We show that there exists a bounded linear operator $T : C^{m,1}(E) \to C^{m,1}(\mathbb{R}^n)$ such that, for any $f \in C^{m,1}(E)$, we have $Tf = f$ on E.

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0. Introduction

Fix $m, n \geq 1$. In [8], we studied the following problems:

Large Finite Problem: Given a finite set $E \subset \mathbb{R}^n$ and functions $f : E \longrightarrow \mathbb{R}$ and $\sigma : E \longrightarrow [0, \infty)$, find the least $M > 0$ for which there exists $F \in C^m(\mathbb{R}^n)$, satisfying $\|F\|_{C^m(\mathbb{R}^n)} \leq M$, and $|F(x) - f(x)| \leq M \cdot \sigma(x)$ for all $x \in E$.

Infinite Problem: Given an arbitrary set $E \subset \mathbb{R}^n$ and functions $f : E \longrightarrow \mathbb{R}$ and $\sigma : E \longrightarrow [0, \infty)$, decide whether there exist a function $F \in C^{m-1,1}(\mathbb{R}^n)$ and a finite constant M, satisfying

(1) $$\|F\|_{C^{m-1,1}(\mathbb{R}^n)} \leq M, \text{ and } |F(x) - f(x)| \leq M \cdot \sigma(x) \text{ for all } x \in E.$$

In the special case $\sigma \equiv 0$, the Infinite Problem amounts to deciding whether a given function $f : E \longrightarrow \mathbb{R}$ extends to a $C^{m-1,1}$ function on all of $\mathbb{R}^n$. (As usual, $C^{m-1,1}(\mathbb{R}^n)$ denotes the space of functions whose $(m-1)^{\underline{\text{rst}}}$ derivatives are Lipschitz 1.) This is a variant of a classical problem of Whitney [18]. Important work on closely related questions was done by Whitney [17,18,19], Glaeser [9], Brudnyi and Shvartsman [2,...,6,12,13,14], and Bierstone-Milman-Pawlucki [1], as explained partially in [8].

In this paper, we will show that an essentially optimal M may be achieved for the two problems above, by taking F to depend linearly on f. More precisely, for the Large Finite Problem, we have the following result.

Theorem 1. *Let $E \subset \mathbb{R}^n$ be finite, and let $\sigma : E \longrightarrow [0, \infty)$ be given. Let $C(E)$ be the vector space of (real-valued) functions on E. Then there exists a linear map $T : C(E) \longrightarrow C^m(\mathbb{R}^n)$, with the following property:*

Let $f \in C(E)$ be given. Assume there exists $F \in C^m(\mathbb{R}^n)$, with $\|F\|_{C^m(\mathbb{R}^n)} \leq 1$ and with $|F(x) - f(x)| \leq \sigma(x)$ for all $x \in E$.

Then we have

$$\|Tf\|_{C^m(\mathbb{R}^n)} \leq A, \text{ and } |Tf(x) - f(x)| \leq A \cdot \sigma(x) \text{ for all } x \in E,$$

for a constant A depending only on m and n.

For the Infinite Problem, we introduce a Banach space $C^{m-1,1}(E,\sigma)$ associated to an arbitrary set $E \subset \mathbb{R}^n$ and a function $\sigma : E \longrightarrow [0,\infty)$. This space consists of all functions $f : E \longrightarrow \mathbb{R}$ for which there exist $F \in C^{m-1,1}(\mathbb{R}^n)$ and $M < \infty$ satisfying (1). The norm $\|f\|_{C^{m-1,1}(E,\sigma)}$ is defined as the infinum of all possible M in (1).

Our result for the Infinite Problem is as follows:

Theorem 2. *Let $E \subset \mathbb{R}^n$ be an arbitrary subset, and let $\sigma : E \longrightarrow [0,\infty)$ be given. Then there exists a linear map $T : C^{m-1,1}(E,\sigma) \longrightarrow C^{m-1,1}(\mathbb{R}^n)$, with the following property:*

Let $f \in C^{m-1,1}(E,\sigma)$ be given with $\|f\|_{C^{m-1,1}(E,\sigma)} \leq 1$.

Then we have

$$\|Tf\|_{C^{m-1,1}(\mathbb{R}^n)} \leq A, \text{ and } |Tf(x) - f(x)| \leq A \cdot \sigma(x) \text{ for all } x \in E$$

for a constant A depending only on m and n.

One of the conjectures of Brudnyi and Shvartsman in [4] is closely analogous to our Theorems 1 and 2. One of their theorems [5] includes the case $\sigma = 0$, $m = 2$ of our results as a special case. I am grateful to Brudnyi and Shvartsman for raising with me the issue of linear dependence of F on f above, and also to E. Bierstone and P. Milman for valuable discussions.

An interesting refinement of Theorem 1 concerns operators of "bounded depth." We say that an operator $T : C(E) \longrightarrow C^m(\mathbb{R}^n)$ has "bounded depth" if every point of $\mathbb{R}^n$ has a neighborhood U, for which $Tf|_U$ depends only on $f|_S$ for a subset $S \subset E$, with $\#(S)$ bounded *a-priori* in terms of m and n. (Here, $\#(S)$ denotes the number of points in S.) The operator T in the conclusion of Theorem 1 may be taken to have bounded depth. This follows without difficulty from our proof of Theorem 1, but we omit the details.

Most of our proof of Theorem 1 repeats ideas in [8] with straightforward modifications. However, we need one additional idea, which we now sketch.

In [8], we introduced the sets

$$\mathcal{K}_f(y;S,C) \;=\; \{J_y(F):\; F\in C^m(\mathbb{R}^n),\; \|F\|_{C^m(\mathbb{R}^n)}\;\le\; C,\; |F(x)-f(x)\,|\;\le\; C\cdot\sigma(x) \text{ on } S\}$$

for $y\in\mathbb{R}^n$, $S\subset E$. Here, and throughout this paper, $J_y(F)$ denotes the $(m-1)$-jet of F at y.

A crucial point in [8] was to show that, for suitable $k^\#$ depending only on m and n, there is an $(m-1)^{\underline{\text{rst}}}$ degree polynomial P belonging to $\mathcal{K}_f(y;S,C)$ for all $S\subset E$ having at most $k^\#$ elements. The set of all such P was called $\mathcal{K}_f(y;k^\#,C)$ in [8].

Roughly speaking, any polynomial in $\mathcal{K}_f(y;k^\#,C)$ is a plausible guess for the $(m-1)$ jet at y of the function Tf in Theorem 1.

To prove Theorem 1, we must not only show that $\mathcal{K}_f(y;k^\#,C)$ is non-empty; we must produce a $P\in\mathcal{K}_f(y;k^\#,C)$ that depends linearly on f. Once this is done, we can essentially repeat the arguments in [8] for large finite sets E and strictly positive σ, because all the functions $F\in C^m(\mathbb{R}^n)$ constructed in [8] depend linearly on f and P.

To find $P\in\mathcal{K}_f(y;k^\#,C)$ depending linearly on f, we introduce the auxiliary convex sets

$$\Gamma(y,S) \;=\; \{J_y(\varphi):\; \|\varphi\|_{C^m(\mathbb{R}^n)}\;\le 1 \text{ and } |\varphi(x)|\;\le\;\sigma(x) \text{ on } S\}\,.$$

By using elementary properties of convex sets, reminiscent of our applications of Helly's theorem in [8], we show that there exists a subset $S^y\subset E$, with the following properties:

(a) The number of points in S^y is bounded by a constant depending only on m and n; and

(b) Any polynomial $P\in\Gamma(y,S^y)$ belongs also to $C\cdot\Gamma(y,S)$, for any $S\subset E$ with at most $k^\#$ points, where C is a constant depending only on m and n. The set S^y depends only on the set E and the function σ, not on f.

Because S^y contains only a few points (property (a) above), it is easy to fit a function $F\in C^m(\mathbb{R}^n)$ to f on S^y, with F depending linearly on f. We may then simply define P to be the $(m-1)$-jet of F at y. Thus, P depends linearly on f. Thanks to property (b) above, we can show also that P belongs to $\mathcal{K}_f(y;k^\#,C)$. This argument appears in Section

10 below, in a lightly disguised form that doesn't explicitly mention F. (The minimization of the quadratic form in Section 10 is morally equivalent to finding F as sketched above, as we see from the standard Whitney extension theorem.)

Once Theorem 1 is established, it isn't hard to deduce Theorem 2. We proceed by applying Theorem 1 to arbitrarily large finite subsets of E, and then passing to a Banach limit. (We recall Banach limits in Section 15 below.)

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We now begin the proofs of Theorems 1 and 2. Unfortunately, we assume from here on that the reader is thoroughly familiar with [8].

1. Notation

Fix $m, n \geq 1$ throughout this paper.

$C^m(\mathbb{R}^n)$ denotes the space of functions $F : \mathbb{R}^n \to \mathbb{R}$ whose derivatives of order $\leq m$ are continuous and bounded on $\mathbb{R}^n$. For $F \in C^m(\mathbb{R}^n)$, we define $\|F\|_{C^m(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} \max_{|\beta|\leq m} |\partial^\beta F(x)|$, and $\|\partial^m F\|_{C^0(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} \max_{|\beta|=m} |\partial^\beta F(x)|$. For $F \in C^m(\mathbb{R}^n)$ and $y \in \mathbb{R}^n$, we define $J_y(F)$ to be the $(m-1)$ jet of F at y, i.e., the polynomial

$$J_y(F)(x) = \sum_{|\beta|\leq m-1} \frac{1}{\beta!}\big(\partial^\beta F(y)\big) \cdot (x-y)^\beta.$$

$C^{m-1,1}(\mathbb{R}^n)$ denotes the space of all functions $F : \mathbb{R}^n \to \mathbb{R}$, whose derivatives of order $\leq m-1$ are continuous, and for which the norm

$$\|F\|_{C^{m-1,1}(\mathbb{R}^n)} = \max_{|\beta|\leq m-1} \left\{ \sup_{x\in\mathbb{R}^n} |\partial^\beta F(x)| + \sup_{\substack{x,y\in\mathbb{R}^n \\ x\neq y}} \frac{|\partial^\beta F(x) - \partial^\beta F(y)|}{|x-y|} \right\}$$

is finite.

Let $\mathcal{P}$ denote the vector space of polynomials of degree $\leq m-1$ on $\mathbb{R}^n$ (with real coefficients), and let D denote the dimension of $\mathcal{P}$.

Let $\mathcal{M}$ denote the set of all multi–indices $\beta = (\beta_1, \ldots, \beta_n)$ with $|\beta| = \beta_1 + \cdots + \beta_n \leq m-1$.

Let $\mathcal{M}^+$ denote the set of multi–indices $\beta = (\beta_1, \dots, \beta_n)$ with $|\beta| \le m$.

If α and β are multi–indices, then $\delta_{\beta\alpha}$ denotes the Kronecker delta, equal to 1 if $\beta = \alpha$ and 0 otherwise.

We will be dealing with functions of x parametrized by y ($x, y \in \mathbb{R}^n$). We will often denote these by $\varphi^y(x)$, or by $P^y(x)$ in case $x \mapsto P^y(x)$ is a polynomial for fixed y. When we write $\partial^\beta P^y(y)$, we always mean the value of $\left(\frac{\partial}{\partial x}\right)^\beta P^y(x)$ at $x = y$; we never use $\partial^\beta P^y(y)$ to denote the derivative of order β of the function $y \mapsto P^y(y)$.

We write $B(x, r)$ to denote the ball with center x and radius r in $\mathbb{R}^n$. If Q is a cube in $\mathbb{R}^n$, then δ_Q denotes the diameter of Q; and $Q^\star$ denotes the cube whose center is that of Q, and whose diameter is 3 times that of Q.

If Q is a cube in $\mathbb{R}^n$, then to "bisect" Q is to partition it into 2^n congruent subcubes in the obvious way. Later on, we will fix a cube $Q^\circ \subset \mathbb{R}^n$, and define the class of "dyadic" cubes to consist of Q°, together with all the cubes arising from Q° by repeated bisection. Each dyadic cube Q other than Q° arises from bisecting a dyadic cube $Q^+ \subseteq Q^\circ$, with $\delta_{Q^+} = 2\delta_Q$. We call Q^+ the dyadic "parent" of Q. Note that $Q^+ \subset Q^\star$.

For any finite set X, write $\#(X)$ to denote the number of elements of X. If X is infinite, then we define $\#(X) = \infty$.

Let $E \subset \mathbb{R}^n$ and $\sigma : E \longrightarrow [0, \infty)$ be given.

Then, as in the Introduction, $C^{m-1,1}(E, \sigma)$ denotes the space of all functions $f : E \longrightarrow \mathbb{R}$, for which there exist $M > 0$, $F \in C^{m-1,1}(\mathbb{R}^n)$, with

(a) $\|F\|_{C^{m-1,1}(\mathbb{R}^n)} \le M$ and

(b) $|F(x) - f(x)| \le M\sigma(x)$ for all $x \in E$.

The norm $\|f\|_{C^{m-1,1}(E,\sigma)}$ is defined as the infinum of the set of all $M > 0$ for which there exists an F satisfying (a) and (b).

Similarly, $C^m(E, \sigma)$ denotes the space of all functions $f : E \longrightarrow \mathbb{R}$, for which there exist $M > 0$, $F \in C^m(\mathbb{R}^n)$, with

(c) $\|F\|_{C^m(\mathbb{R}^n)} \leq M$ and

(d) $|F(x) - f(x)| \leq M\sigma(x)$ for all $x \in E$.

The norm $\|f\|_{C^m(E,\sigma)}$ is defined as inf of all $M > 0$ for which there exists F satisfying (c) and (d).

Suppose $E \subset \mathbb{R}^n$ (finite), $\sigma : E \longrightarrow (0, \infty)$, and $\delta > 0$ are given. If $f : E \longrightarrow \mathbb{R}$, then the norm $\|f\|_{C^m(E,\sigma;\delta)}$ is defined as the inf. of all $M > 0$ for which there exists $F \in C^m(\mathbb{R}^n)$, with $\|\partial^\beta F\|_{C^0(\mathbb{R}^n)} \leq M\delta^{-|\beta|}$ for $|\beta| \leq m$, and

$$|F(x) - f(x)| \leq M\delta^{-m}\sigma(x) \text{ for all } x \in E\,.$$

We write $C^m(E, \sigma; \delta)$ for the space of all functions $f : E \longrightarrow \mathbb{R}$, equipped with the above norm.

If $\delta > 0$ and $F \in C^m(\mathbb{R}^n)$, then we define

$$\|F\|_{C^m(\mathbb{R}^n;\delta)} = \max_{|\beta| \leq m} \delta^{|\beta|} \|\partial^\beta F\|_{C^0(\mathbb{R}^n)}\,.$$

We write $C^m(\mathbb{R}^n; \delta)$ for the space $C^m(\mathbb{R}^n)$, equipped with the norm $\|F\|_{C^m(\mathbb{R}^n;\delta)}$.

If $E \subset \mathbb{R}^n$ is finite, then $C^0(E)$ denotes the space of functions $f : E \longrightarrow \mathbb{R}$, equipped with the norm $\|f\|_{C^0(E)} = \max_{x \in E} |f(x)|$.

A subset $\mathcal{K} \subset \mathbb{R}^d$ is called *symmetric* if, for any $x \in \mathbb{R}^d$, $x \in \mathcal{K}$ implies $-x \in \mathcal{K}$.

If $\mathcal{K} \subset \mathbb{R}^d$ is symmetric, and if $C > 0$ is given then $C\mathcal{K}$ denotes the set of all the points $Cx \in \mathbb{R}^d$ with $x \in \mathcal{K}$.

2. Sharp Whitney

One of the main results of [8] is as follows.

Sharp Whitney Theorem for Finite Sets:

Given $m, n \geq 1$, there exist constants $k^\#_{sw}(m, n)$ and $A(m, n)$, depending only on m and n, for which the following holds.

Let $E \subset \mathbb{R}^n$ be finite, and let $f : E \longrightarrow \mathbb{R}$, $\sigma : E \longrightarrow [0, \infty)$ be functions on E. Assume that, given any $S \subset E$ with $\#(S) \leq k^{\#}_{sw}(m,n)$, there exists $F^S \in C^m(\mathbb{R}^n)$, with $\|F^S\|_{C^m(\mathbb{R}^n)} \leq 1$, and $|F^S(x) - f(x)| \leq \sigma(x)$ for all $x \in S$.

Then there exists $F \in C^m(\mathbb{R}^n)$, with

$$\|F\|_{C^m(\mathbb{R}^n)} \leq A(m,n)\,, \text{ and } |F(x) - f(x)| \leq A(m,n) \cdot \sigma(x) \text{ for all } x \in E\,.$$

In terms of the spaces $C^m(E, \sigma; \delta)$, we have the following

COROLLARY: *Given $m, n \geq 1$, there exist constants $k^{\#}_{sw}(m,n)$ and $A(m,n)$, depending only on m and n, for which the following holds.*

Let $E \subset \mathbb{R}^n$ be finite, let $f : E \longrightarrow \mathbb{R}$, let $\sigma : E \longrightarrow [0, \infty)$, and let $\delta > 0$. Then

$$\|f\|_{C^m(E,\sigma;\delta)} \leq A(m,n) \cdot \max\{\|f\|_{C^m(S,\sigma|_S;\delta)} : S \subset E, \#(S) \leq k^{\#}_{sw}(m,n)\}$$

Proof: The case $\delta = 1$ is immediate from Sharp Whitney for Finite Sets; the general case follows by rescaling. ■

3. A LEMMA ON CONVEX SETS

The following result is surely known (probably in sharper form) but I haven't found it in the literature.

Lemma on Convex Sets:

Let $\mathcal{F}$ be a finite collection of compact, convex, symmetric subsets of $\mathbb{R}^D$. Suppose 0 is an interior point of each $\mathcal{K} \in \mathcal{F}$. Then, with C_D depending only on D, and with $\ell = D \cdot (D+1)$, there exist $\mathcal{K}_1, \cdots, \mathcal{K}_\ell \in \mathcal{F}$, with

$$\mathcal{K}_1 \cap \cdots \cap \mathcal{K}_\ell \subset C_D \cdot \left(\bigcap_{\mathcal{K} \in \mathcal{F}} \mathcal{K} \right).$$

Proof: We use the following standard results on convex sets.

Helly's Theorem [16]: *Let $\mathcal{F}$ be any collection of compact, convex subsets of $\mathbb{R}^D$. If the intersection of all the sets $\mathcal{K} \in \mathcal{F}$ is empty, then already the intersection of some $(D+1)$ sets $\mathcal{K}_1, \cdots, \mathcal{K}_{D+1} \in \mathcal{F}$ is empty.*

F. John Lemma (See the simple proof by A. Cordoba in [10]): *Let $\mathcal{K} \subset \mathbb{R}^D$ be any bounded, symmetric, convex set with non-empty interior. Then there exists an ellipsoid $E \subset \mathbb{R}^D$, centered at the origin, with $E \subset \mathcal{K} \subset C_D E$, where C_D depends only on D.*

Proof of the Lemma on Convex Sets: Let $\mathcal{K}^*$ be the intersection of all the sets $\mathcal{K} \in \mathcal{F}$. Then $\mathcal{K}^*$ is compact, convex, symmetric, and contains 0 as an interior point. Applying the Lemma of F. John, we obtain an ellipsoid $E \subset \mathbb{R}^D$, centered at the origin, with $E \subset \mathcal{K}^* \subset C_D \cdot E$. Applying a linear transformation to $\mathbb{R}^D$, we may assume without loss of generality that E is the unit ball. Hence, for constants c_D, C'_D, depending only on D, we have

$$(1) \quad c_D Q \subset \mathcal{K}^* \subset C'_D Q, \quad \text{with} \\ Q = \{x = (x_1, \cdots, x_D) \in \mathbb{R}^D : |x_j| < 1 \text{ for each } j\}.$$

Given $\mathcal{K} \in \mathcal{F}$ and $1 \leq j \leq D$, we set

$$\mathrm{Cap}(\mathcal{K}, j) = \{(x_1, \cdots, x_D) \in \mathcal{K} : x_j \geq C'_D\}.$$

Each $\mathrm{Cap}(\mathcal{K}, j)$ is a compact, convex subset of $\mathbb{R}^D$. Moreover, since $\mathcal{K}^* \subset C'_D Q$, we know that the intersection of all the sets $\mathrm{Cap}(\mathcal{K}, j)$ ($\mathcal{K} \in \mathcal{F}$) is empty, for each fixed j. Applying Helly's Theorem to the $\mathrm{Cap}(\mathcal{K}, j)$, we obtain sets $\mathcal{K}_1^{(j)}, \ldots, \mathcal{K}_{D+1}^{(j)} \in \mathcal{F}$, for which the intersection of $\mathrm{Cap}(\mathcal{K}_i^{(j)}, j)$ over $i = 1, \cdots, D+1$ is empty. This means that every $x = (x_1, \cdots, x_D) \in \mathcal{K}_1^{(j)} \cap \cdots \cap \mathcal{K}_{D+1}^{(j)}$ satisfies $x_j < C'_D$. Since the $\mathcal{K}_i^{(j)}$ are symmetric, we have $|x_j| < C'_D$ for all $x = (x_1, \cdots, x_D) \in \bigcap_{i=1}^{D+1} \mathcal{K}_i^{(j)}$. Consequently, the intersection $\bigcap_{j=1}^{D} \bigcap_{i=1}^{D+1} \mathcal{K}_i^{(j)}$ is contained in $C'_D Q$, which in turn is contained in $\left(C'_D / c_D\right) \cdot \bigcap_{\mathcal{K} \in \mathcal{F}} \mathcal{K}$, thanks to (1). Thus, we have found $D \cdot (D+1)$ sets $\mathcal{K}_i^{(j)} \in \mathcal{F}$, whose intersection is contained in $C''_D \cdot \bigcap_{\mathcal{K} \in \mathcal{F}} \mathcal{K}$. The proof of the Lemma is complete. ■

4. Statement of the Main Lemmas

Fix $\mathcal{A} \subset \mathcal{M}$. We state two results involving $\mathcal{A}$. For the second result, we use an order relation between multi-indices, defined in [8] and denoted by $>$.

Weak Main Lemma for $\mathcal{A}$. *Given $m, n \geq 1$, there exist constants $k^{\#}, a_0$, depending only on m and n, for which the following holds.*

Suppose we are given a finite set $E \subset \mathbb{R}^n$ and a function $\sigma : E \longrightarrow (0, \infty)$. Suppose we are also given a point $y^0 \in \mathbb{R}^n$ and a family of polynomials $P_\alpha \in \mathcal{P}$, indexed by $\alpha \in \mathcal{A}$. Assume that the following conditions are satisfied:

(WL1) $\partial^\beta P_\alpha(y^0) = \delta_{\beta\alpha}$ for all $\beta, \alpha \in \mathcal{A}$.

(WL2) $|\partial^\beta P_\alpha(y^0) - \delta_{\beta\alpha}| \leq a_0$ for all $\alpha \in \mathcal{A}, \beta \in \mathcal{M}$.

(WL3) Given $S \subset E$ with $\#(S) \leq k^{\#}$, and given $\alpha \in \mathcal{A}$, there exists $\varphi_\alpha^S \in C^m(\mathbb{R}^n)$, with

(a) $\|\partial^m \varphi_\alpha^S\|_{C^0(\mathbb{R}^n)} \leq a_0$.

(b) $|\varphi_\alpha^S(x)| \leq C\sigma(x)$ for all $x \in S$.

(c) $J_{y^0}(\varphi_\alpha^S) = P_\alpha$.

Then there exists a linear operator $\mathcal{E} : C^m(E, \sigma) \longrightarrow C^m(\mathbb{R}^n)$, satisfying the following conditions:

(WL4) $\mathcal{E}$ has norm at most C';

(WL5) $|\mathcal{E}f(x) - f(x)| \leq C' \cdot \|f\|_{C^m(E,\sigma)} \cdot \sigma(x)$ for all $f \in C^m(E, \sigma)$ and $x \in E \cap B(y^0, c')$. Here, c' and C' in **(WL4,5)** depend only on C, m, n in **(WL1,2,3)**.

Strong Main Lemma for $\mathcal{A}$. *Given $m, n \geq 1$, there exists $k^{\#}$, depending only on m and n, for which the following holds.*

Suppose we are given a finite set $E \subset \mathbb{R}^n$, and a function $\sigma : E \longrightarrow (0, \infty)$.

Suppose we are also given a point $y^0 \in \mathbb{R}^n$, and a family of polynomials $P_\alpha \in \mathcal{P}$, indexed by $\alpha \in \mathcal{A}$. Assume that the following conditions are satisfied:

(SL1) $\partial^\beta P_\alpha(y^0) = \delta_{\beta\alpha}$ for all $\alpha, \beta \in \mathcal{A}$.

(SL2) $|\partial^\beta P_\alpha(y^0)| \leq C$ for all $\alpha \in \mathcal{A}, \beta \in \mathcal{M}$ with $\beta \geq \alpha$.

(SL3) Given $S \subset E$ with $\#(S) \leq k^\#$, and given $\alpha \in \mathcal{A}$, there exists $\varphi_\alpha^S \in C^m(\mathbb{R}^n)$, with

(a) $\|\partial^m \varphi_\alpha^S\|_{C^0(\mathbb{R}^n)} \leq C$,

(b) $|\varphi_\alpha^S(x)| \leq C\sigma(x)$ for all $x \in S$,

(c) $J_{y^0}(\varphi_\alpha^S) = P_\alpha$.

Then there exists a linear operator $\mathcal{E} : C^m(E,\sigma) \longrightarrow C^m(\mathbb{R}^n)$, satisfying the following conditions:

(SL4) $\mathcal{E}$ has norm at most C';

(SL5) $|\mathcal{E}f(x) - f(x)| \leq C'\|f\|_{C^m(E,\sigma)} \cdot \sigma(x)$ for all $f \in C^m(E,\sigma)$ and all $x \in E \cap B(y^0, c')$. Here, c' and C' in **(SL4,5)** depend only on C, m, n in **(SL1,2,3)**.

5. Plan of the Proof

Recall from [8] that subsets of $\mathcal{M}$ are totally ordered by a relation denoted by $<$.

As in [8], we will establish the WEAK and STRONG MAIN LEMMAS for any $\mathcal{A} \subset \mathcal{M}$, by proving the following results.

Lemma PP1: *The* WEAK *and* STRONG MAIN LEMMAS *both hold for* $\mathcal{A} = \mathcal{M}$.

Lemma PP2: *Fix* $\mathcal{A} \subset \mathcal{M}$, *with* $\mathcal{A} \neq \mathcal{M}$. *Assume that the* STRONG MAIN LEMMA *holds for each* $\bar{\mathcal{A}} < \mathcal{A}$. *Then the* WEAK MAIN LEMMA *holds for* $\mathcal{A}$.

Lemma PP3: *Fix* $\mathcal{A} \subseteq \mathcal{M}$, *and assume that the* WEAK MAIN LEMMA *holds for all* $\bar{\mathcal{A}} \leq \mathcal{A}$. *Then the* STRONG MAIN LEMMA *holds for* $\mathcal{A}$.

Once we have established these three Lemmas, the two MAIN LEMMAS must hold for all $\mathcal{A}$, by induction on $\mathcal{A}$. Taking $\mathcal{A}$ to be the empty set in (say) the WEAK MAIN LEMMA, we see that hypotheses **(WL1,2,3)** hold vacuously; hence we obtain the following result.

Local Theorem 1: *Given $m, n \geq 1$, there exist A, $c' > 0$, depending only on m and n, for which the following holds.*

Let $E \subset \mathbb{R}^n$ be finite, and let $\sigma : E \longrightarrow (0, \infty)$ be given. Let $y^0 \in \mathbb{R}^n$. Then there exists a linear operator $\mathcal{E} : C^m(E, \sigma) \longrightarrow C^m(\mathbb{R}^n)$, with norm at most A, and satisfying

$$|\mathcal{E}f(x) - f(x)| \leq A \|f\|_{C^m(E,\sigma)} \cdot \sigma(x) \text{ for all } f \in C^m(E, \sigma) \text{ and all } x \in E \cap B(y^0, c') .$$

We will then relax the hypothesis $\sigma : E \longrightarrow (0, \infty)$ to $\sigma : E \longrightarrow [0, \infty)$, and next deduce Theorem 1 by using an obvious partition of unity. Finally, we deduce Theorem 2 from Theorem 1. These arguments are given in sections 14,...,17.

6. Starting the Main Induction

In this section, we prove Lemma PP1. That is, we prove the two MAIN LEMMAS for $\mathcal{A} = \mathcal{M}$. We simply take $\mathcal{E} = 0$, and assume either **(WL1,2,3)** or **(SL1,2,3)**, for our given E, σ.

Suppose $\|f\|_{C^m(E,\sigma)} \leq 1$. Then there exists $F \in C^m(\mathbb{R}^n)$ with

$$\|F\|_{C^m(\mathbb{R}^n)} \leq 2 \text{ and } |F(x) - f(x)| \leq 2\sigma(x) \text{ on } E .$$

Hence, $|f(x)| \leq 2 + 2\sigma(x)$ on E. On the other hand, the proof of (6.2) in Section 6 of [8] applies here, and shows that $\sigma(x) \geq \frac{1}{2C}$ for all $x \in E \cap B(y^0, c')$, with C as in **(WL1,2,3)** or **(SL1,2,3)**, and with c' determined by C, m, n. Consequently,

$$|\mathcal{E}f(x) - f(x)| = |f(x)| \leq (4C + 2) \cdot \sigma(x) \text{ for all } x \text{ in } E \cap B(y^0, c') .$$

This holds provided $\|f\|_{C^m(E,\sigma)} \leq 1$. The conclusions of the two MAIN LEMMAS are now obvious. ■

7. Non-Monotonic Sets

In this section, we prove Lemma PP2 for non-monotonic $\mathcal{A}$.

Lemma NMS: *Fix a non-monotonic set $\mathcal{A} \subset \mathcal{M}$, and assume that the* STRONG MAIN LEMMA *holds for all $\bar{\mathcal{A}} < \mathcal{A}$. Then the* WEAK MAIN LEMMA *holds for $\mathcal{A}$.*

Proof: Let E, σ satisfy **(WL1,2,3)** for $\mathcal{A}$. Since $\mathcal{A}$ is not monotonic, there exist multi-indices $\bar{\alpha}, \bar{\gamma}$, with $\bar{\alpha} \in \mathcal{A}$, $\bar{\alpha} + \bar{\gamma} \in \mathcal{M} \backslash \mathcal{A}$. We set $\bar{\mathcal{A}} = \mathcal{A} \cup \{\bar{\alpha} + \bar{\gamma}\}$. As in the proof of Lemma 7.1 in [8], we see that $\bar{\mathcal{A}} < \mathcal{A}$, and that the hypotheses **(SL1,2,3)** of the STRONG MAIN LEMMA hold for $\bar{\mathcal{A}}$, with constants depending only on C, m, n in **(WL1,2,3)** for $\mathcal{A}$.

Applying the STRONG MAIN LEMMA for $\bar{\mathcal{A}}$, we obtain a linear operator $\mathcal{E} : C^m(E, \sigma) \longrightarrow C^m(\mathbb{R}^n)$, satisfying **(SL4,5)**, with constants C', c' depending only on C, m, n in **(WL1,2,3)** for $\mathcal{A}$. However, **(SL4,5)** are the same as the conclusions **(WL4,5)** of the WEAK MAIN LEMMA for $\mathcal{A}$. The proof of Lemma NMS is complete. ■

8. A Consequence of the Main Inductive Assumption

In this section, we establish the following result.

Lemma CMIA: *Fix $\mathcal{A} \subset \mathcal{M}$, and assume that the* STRONG MAIN LEMMA *holds, for all $\bar{\mathcal{A}} < \mathcal{A}$. Then there exists $k^{\#}_{\text{old}} \geq k^{\#}_{sw}(m,n)$, depending only on m and n, for which the following holds.*

Let $A > 0$ be given, let $Q \in \mathbb{R}^n$ be a cube, $\hat{E} \subset \mathbb{R}^n$ a finite set, $\sigma : \hat{E} \longrightarrow (0, \infty)$ a function. Suppose that, for each $y \in Q^{**}$, we are given a set $\bar{\mathcal{A}}^y < \mathcal{A}$ and a family of polynomials $\bar{P}^y_\alpha \in \mathcal{P}$, indexed by $\alpha \in \bar{\mathcal{A}}^y$. Assume that the following conditions are satisfied:

(G1) $\partial^\beta \bar{P}^y_\alpha(y) = \delta_{\beta\alpha}$ for all $\beta, \alpha \in \bar{\mathcal{A}}^y$, $y \in Q^{**}$

(G2) $|\partial^\beta \bar{P}^y_\alpha(y)| \leq A\delta_Q^{|\alpha|-|\beta|}$ for all $\alpha \in \bar{\mathcal{A}}^y$, $\beta \geq \alpha$, $y \in Q^{**}$.

(G3) Given $S \subset \hat{E}$ with $\#(S) \leq k^{\#}_{\text{old}}$, and given $y \in Q^{**}$ and $\alpha \in \bar{\mathcal{A}}^y$, there exists $\varphi^S_\alpha \in C^m(\mathbb{R}^n)$, with

(a) $\|\partial^m \varphi^S_\alpha\|_{C^0(\mathbb{R}^n)} \leq A \cdot \delta_Q^{|\alpha|-m}$,

(b) $|\varphi^S_\alpha(x)| \leq A \cdot \delta_Q^{|\alpha|-m} \cdot \sigma(x)$ for all $x \in S$,

(c) $J_y(\varphi^S_\alpha) = \bar{P}^y_\alpha$

Then there exists $\mathcal{E} : C^m(\hat{E}, \sigma; \delta_Q) \longrightarrow C^m(\mathbb{R}^n; \delta_Q)$, with the following properties:

(G4) $\mathcal{E}$ has norm at most A';

(G5) $|\mathcal{E}f(x)-f(x)| \le A'\delta_Q^{-m}\|f\|_{C^m(\hat{E},\sigma;\delta_Q)}\cdot\sigma(x)$ for all $f \in C^m(\hat{E},\sigma;\delta_Q)$ and all $x \in \hat{E}\cap Q^*$.

Here, A' depends only on A, m, n.

Proof: As in Section 8 of [8], a rescaling reduces matters to the case $\delta_Q = 1$.

Let $\delta_Q = 1$, and assume (**G1,2,3**). For each $y \in Q^{**}$, the hypotheses (**SL1,2,3**) of the STRONG MAIN LEMMA for $\bar{\mathcal{A}}^y$ hold, with $\hat{E}, \sigma, y, \bar{P}_\alpha^y(\alpha \in \bar{\mathcal{A}}^y)$, A in place of $E, \sigma, y^0, P_\alpha(\alpha \in \mathcal{A})$, C in (**SL1,2,3**). In fact, (**SL1,2,3**) are immediate from (**G1,2,3**), provided we take $k^{\#}_{\rm old}$ to be the max of $k^{\#}_{sw}(m,n)$ and the constants $k^{\#}$ appearing in the strong main lemma for all $\bar{\mathcal{A}} < \mathcal{A}$. Hence, the STRONG MAIN LEMMA for $\bar{\mathcal{A}}^y$ produces an operator $\mathcal{E}_y : C^m(\hat{E},\sigma) \longrightarrow C^m(\mathbb{R}^n)$, satisfying

(1) $\quad \|\mathcal{E}_y\| \le A'$; and

(2) $$\left[\begin{array}{l} |\mathcal{E}_y f(x) - f(x)| \le A' \cdot \|f\|_{C^m(\hat{E},\sigma)} \cdot \sigma(x) \\ \\ \text{for all } f \in C^m(\hat{E},\sigma),\ x \in \hat{E}\cap B(y,c'),\ y \in Q^{**}\,. \end{array}\right]$$

Here, A' and c' are determined by A, m, n in (**G1,2,3**).

Given $f \in C^m(\hat{E},\sigma)$, we set $F^y = \mathcal{E}_y f$ for each $y \in Q^{**}$. We then define F from the F^y as in (8.1)-(8.9) in [8]. Since $F \in C^m(\mathbb{R}^n)$ depends linearly on the F^y, which depend linearly on f, the map $\mathcal{E} : f \longrightarrow F$ is a linear operator from $C^m(\hat{E},\sigma)$ to $C^m(\mathbb{R}^n)$. Moreover, if $\|f\|_{C^m(\hat{E},\sigma)} \le 1$, then (1) and (2) show that

$$\|F^y\|_{C^m(\mathbb{R}^n)} \le A' \text{ and } |F^y(x) - f(x)| \le A'\cdot\sigma(x) \text{ for all } x \in \hat{E}\cap B(y,c')\,,$$

Hence, the proof of (8.8) and (8.9) in [8] goes through here, and we have $\|F\|_{C^m(\mathbb{R}^n)} \le C''$ with C'' determined by A, m, n; and $|F(x) - f(x)| \le A'\cdot\sigma(x)$ for all $x \in \hat{E}\cap Q^*$. That is,

(3) $\quad \|\mathcal{E}f\|_{C^m(\mathbb{R}^n)} \le C''$ if $\|f\|_{C^m(\hat{E},\sigma)} \le 1$; and

(4) $\quad |\mathcal{E}f(x) - f(x)| \le A'\cdot\sigma(x)$ for all $x \in \hat{E}\cap Q^*$, provided $\|f\|_{C^m(\hat{E},\sigma)} \le 1$.

From (3) and (4) we obtain trivially the desired conclusions (**G4,5**) in the case $\delta_Q = 1$. The proof of the Lemma is complete. ■

9. SET-UP FOR THE MAIN INDUCTION

In this section, we give the set-up for the proof of Lemma PP2 in the monotonic case. We fix $m, n \geq 1$ and $\mathcal{A} \subset \mathcal{M}$. We let $k^{\#}$ be a large enough integer determined by m and n, to be picked later. We suppose we are given $E \subset \mathbb{R}^n$ finite, $\sigma : E \longrightarrow (0, \infty)$, $y^0 \in \mathbb{R}^n$, $P_\alpha \in \mathcal{P}$ indexed by $\alpha \in \mathcal{A}$. In addition, we suppose we are given a positive number a_1. We fix $k^{\#}$, E, σ, y^0, $(P_\alpha)_{\alpha \in \mathcal{A}}, a_1$ until the end of Section 12. We make the following assumptions.

(**SU0**) $\mathcal{A}$ is monotonic, and $\mathcal{A} \neq \mathcal{M}$.

(**SU1**) The STRONG MAIN LEMMA holds for all $\bar{\mathcal{A}} < \mathcal{A}$.

(**SU2**) $\partial^\beta P_\alpha(y^0) = \delta_{\beta\alpha}$ for all $\beta, \alpha \in \mathcal{A}$.

(**SU3**) $|\partial^\beta P_\alpha(y^0) - \delta_{\beta\alpha}| \leq a_1$ for all $\alpha \in \mathcal{A}, \beta \in \mathcal{M}$.

(**SU4**) a_1 is less than a small enough constant determined by m and n.

(**SU5**) Given $S \subset E$ with $\#(S) \leq k^{\#}$, and given $\alpha \in \mathcal{A}$, there exists $\varphi^S_\alpha \in C^m(\mathbb{R}^n)$, with

(a) $\|\partial^m \varphi^S_\alpha\|_{C^0(\mathbb{R}^n)} \leq a_1$,

(b) $|\varphi^S_\alpha(x)| \leq \sigma(x)$ for all $x \in S$,

(c) $J_{y^0}(\varphi^S_\alpha) = P_\alpha$.

Most of the effort of this paper goes into proving the following result.

<u>**Lemma SU.I**</u>: Assume (**SU0,...,5**). Then there exists a linear operator $\mathcal{E} : C^m(E, \sigma) \longrightarrow C^m(\mathbb{R}^n)$, satisfying

(a) $\|\mathcal{E}\| \leq A$, and

(b) $|\mathcal{E}f(x) - f(x)| \leq A\|f\|_{C^m(E,\sigma)} \cdot \sigma(x)$ for all $f \in C^m(E, \sigma)$ and all $x \in E \cap B(y^0, a)$.

Here, A and a are determined by a_1, m, n.

Note that a_1 is not assumed to depend only on m and n, and that the constant C in (**WL3**)(b) has in effect been set equal to 1 in (**SU5**)(b).

The following result is trivial. (Compare with Lemma 9.2 in [8].)

Lemma SU.II: Lemma SU.I implies Lemma PP2.

10. Applying Lemmas on Convex Sets

We place ourselves in the setting of Section 9, and we assume (**SU0,..., 5**). Recall that we have fixed $E \subset \mathbb{R}^n$ finite, and $\sigma : E \longrightarrow (0, \infty)$.

For $y \in \mathbb{R}^n$ and $S \subset E$, we set

(1) $$\Gamma(y, S) = \{J_y(\varphi) : \|\varphi\|_{C^m(\mathbb{R}^n)} \leq 1, \text{ and } |\varphi(x)| \leq \sigma(x) \text{ on } S\} \subseteq \mathcal{P}\,.$$

Each $\Gamma(y, S)$ is bounded, convex, symmetric, and contains 0 as an interior point. Moreover, there are only finitely many subsets $S \subset E$. Hence, we may apply the LEMMA ON CONVEX SETS to the closures of the $\Gamma(y, S)$ for any fixed y, and $S \subset E$ arbitrary, subject to $\#(S) \leq k^\#$. Thus, we obtain subsets $S_1, \ldots, S_{D\cdot(D+1)} \subset E$, with $\#(S_i) \leq k^\#$ for each i, and satisfying the following inclusion:

$$\bigcap_{i=1}^{D(D+1)} \Gamma(y, S_i) \subset C_D \cdot \bigcap_{\substack{S \subset E \\ \#(S) \leq k^\#}} \text{Closure}\,(\Gamma(y, S))\,.$$

Moreover, Closure $(\Gamma(y, S)) \subset 2\Gamma(y, S)$, since 0 is an interior point of the convex set $\Gamma(y, S)$. Hence, with $C'_D = 2C_D$, we have

(2) $$\bigcap_{i=1}^{D\cdot(D+1)} \Gamma(y, S_i) \subset C'_D \cdot \bigcap_{\substack{S \subset E \\ \#(S) \leq k^\#}} \Gamma(y, S)\,.$$

Here, C'_D depends only on D, and of course the S_i depend on y. Let

(3) $$S^y = S_1 \cup \cdots \cup S_{D(D+1)}\,.$$

Then, obviously,

(4) $S^y \subset E$

(5) $\#(S^y) \leq D \cdot (D+1) \cdot k^{\#}$,

and

$\Gamma(y, S^y) \subset \Gamma(y, S_i)$ for $i = 1, \cdots, D(D+1)$, so that (2) implies

(6) $\Gamma(y, S^y) \subset C'_D \cdot \Gamma(y, S)$ for all $S \subset E$ with $\#(S) \leq k^{\#}$.

Next, using S^y, we introduce a linear operator

(7) $T^y : C^m(E, \sigma) \longrightarrow \mathcal{P}$.

We proceed as follows. Let $S^y \cup \{y\} = \{x_1, \ldots, x_N\}$, with $x_N = y$. We introduce the vector space $\mathcal{P}^N$ of all

(8) $\vec{P} = (P_\mu)_{1 \leq \mu \leq N}$ with each $P_\mu \in \mathcal{P}$.

Given a function $f \in C^m(E, \sigma)$, we define a quadratic function $\mathcal{Q}_f^y$ on $\mathcal{P}^N$, by setting

(9)
$$\mathcal{Q}_f^y\left(\vec{P}\right) = \sum_{\mu=1}^{N} \sum_{|\beta| \leq m-1} |\partial^\beta P_\mu(x_\mu)|^2 + \sum_{\mu \neq \nu} \sum_{|\beta| \leq m-1} \frac{|\partial^\beta (P_\mu - P_\nu)(x_\mu)|^2}{|x_\mu - x_\nu|^{2 \cdot (m - |\beta|)}}$$
$$+ \sum_{\mu=1}^{N} 1_{x_\mu \in E} \cdot \frac{|P_\mu(x_\mu) - f(x_\mu)|^2}{(\sigma(x_\mu))^2} \quad \text{for } \vec{P} \quad \text{as in (8).}$$

Here, the characteristic function $1_{x_\mu \in E}$ enters, since we don't know whether y belongs to E.

The quadratic function $\mathcal{Q}_f^y$ contains O^{th}, 1^{st}, and 2^{nd} degree terms in $\vec{P}$. The sum of the second-degree terms is a strictly positive-definite quadratic form, independent of f. Also, the first degree terms are linear in f. It follows that $\vec{P} \mapsto \mathcal{Q}_f^y(\vec{P})$ achieves a minimum at a point $\vec{P}(f, y) \in \mathcal{P}^N$ that depends linearly on f for fixed y. The components of $\vec{P}(f, y)$ may be denoted by $P_\mu(f, y) \in \mathcal{P}$, for $\mu = 1, \ldots, N$. We define T^y in (7) by setting

(10) $T^y f = P_N(f, y)$.

Thus, T^y is a linear operator from $C^m(E,\sigma)$ to $\mathcal{P}$. Note that

(11) $\quad T^y f$ depends only on $f|_{S^y \cup \{y\}}$ if $y \in E$, and only on $f|_{S^y}$ if $y \notin E$.

Next, we prove the following result.

<u>**Lemma 1**</u>: *Given $y \in \mathbb{R}^n$ and $f \in C^m(E,\sigma)$, there exists $\tilde{F} \in C^m(\mathbb{R}^n)$, with*

(12) $\quad \|\tilde{F}\|_{C^m(\mathbb{R}^n)} \leq C\, \|f\|_{C^m(E,\sigma)}$,

(13) $\quad |\tilde{F}(x) - f(x)| \leq C \cdot \sigma(x) \cdot \|f\|_{C^m(E,\sigma)}$ for all $x \in S^y$,

(14) $\quad J_y(\tilde{F}) = T^y f$.

Here, C depends only on m and n.

<u>*Proof*</u>: Throughout the proof, let C denote a constant determined by m and n. Without loss of generality, we may suppose that

(15) $\quad \|f\|_{C^m(E,\sigma)} = 1$.

By definition of the norm in (15), there exists $F \in C^m(\mathbb{R}^n)$, with

(16) $\quad \|F\|_{C^m(\mathbb{R}^n)} \leq 2$, and

(17) $\quad |F(x) - f(x)| \leq 2\sigma(x)$ for all $x \in E$.

Define $\vec{P} = (P_\mu)_{1 \leq \mu \leq N} \in \mathcal{P}^N$ by setting $P_\mu = J_{x_\mu}(F)$. Thus, (16) and (17) imply the following estimates:

$|\partial^\beta P_\mu(x_\mu)| \leq 2$ for $1 \leq \mu \leq N$, $|\beta| \leq m-1$.

$|\partial^\beta (P_\mu - P_\nu)(x_\mu)| \leq C|x_\mu - x_\nu|^{m-|\beta|}$ for $\mu \neq \nu$, $|\beta| \leq m-1$.

$|P_\mu(x_\mu) - f(x_\mu)| \leq 2\sigma(x_\mu)$ if $x_\mu \in E$.

Hence, for this $\vec{P}$, each summand in (9) is at most C. Moreover, (5) shows that the number of summands in (9) is at most C. Consequently, $\mathcal{Q}_f^y(\vec{P}) \leq C$. Since $\vec{P}(f,y)$ was picked to minimize $\mathcal{Q}_f^y$, we conclude that $\mathcal{Q}_f^y(\vec{P}(f,y)) \leq C$. In particular, we have

(18) $|\partial^\beta [P_\mu(f,y)]\,(x_\mu)| \,\le\, C$ for $1 \,\le\, \mu \,\le\, N\,,\ |\beta| \,\le\, m-1\,,$

(19) $|\partial^\beta [P_\mu(f,y) - P_\nu(f,y)]\,(x_\mu)| \,\le\, C \,\cdot\, |x_\mu - x_\nu|^{m-|\beta|}$ for $\mu \neq \nu$, $|\beta| \,\le\, m-1\,,$

(20) $|[P_\mu(f,y)](x_\mu) - f(x_\mu)| \,\le\, C\,\sigma(x_\mu)$ if $x_\mu \in E\,.$

By the standard Whitney extension theorem (see [11] or [15]) and (18), (19), there exists a function $\tilde{F} \in C^m(\mathbb{R}^n)$, with

(21) $\|\tilde{F}\|_{C^m(\mathbb{R}^n)} \,\le\, C\,,$ and

(22) $J_{x_\mu}(\tilde{F}) \,=\, P_\mu(f,y)$ for each $\mu = 1, \ldots N\,.$

By (20), (22), and the definition of $x_1, \cdots, x_N$, we have

(23) $|\tilde{F}(x) - f(x)| \,\le\, C\,\sigma(x)$ for all $x \in S^y\,.$

Moreover, (10) and (22) yield

(24) $J_y(\tilde{F}) \,=\, T^y f\,,$

since $x_N = y$.

Under our assumption (15), results (21), (23), (24) are precisely the conclusions (12), (13), (14) of Lemma 1.

The proof of the lemma is complete. ■

For $y \in \mathbb{R}^n$, $f \in C^m(E,\sigma)$, $M > 0$, we define

$$\mathcal{K}_f(y, k^\#, M) = \{P \in \mathcal{P}:\ \text{Given } S \subset E \text{ with } \#(S) \le k^\#,\ \text{there exists } F^S \in C^m(\mathbb{R}^n),$$

$$\text{with } \|F^S\|_{C^m(\mathbb{R}^n)} \le M, |F^S(x) - f(x)| \le M \cdot \sigma(x) \text{ for all } x \in S, \text{ and } J_y(F^S) = P\},$$ as in [8].

Lemma 2: *Given $y \in \mathbb{R}^n$ and $f \in C^m(E,\sigma)$ with $\|f\|_{C^m(E,\sigma)} \le 1$, we have*

$$T^y f \in \mathcal{K}_f(y, k^\#, C)$$

for a large enough C determined by m and n.

Proof: Let $\tilde{F}$ be as in Lemma 1. Since $\|f\|_{C^m(E,\sigma)} \leq 1$, we have

(25) $$\|\tilde{F}\|_{C^m(\mathbb{R}^n)} \leq C\,,$$

(26) $$|\tilde{F}(x) - f(x)| \leq C \cdot \sigma(x) \text{ on } S^y, \text{ and}$$

(27) $$J_y(\tilde{F}) = T^y f;$$

throughout the proof of Lemma 2, C, C', c, etc. denote constants determined by m and n. Also, since $\|f\|_{C^m(E,\sigma)} \leq 1$, there exists $F \in C^m(\mathbb{R}^n)$, with

(28) $$\|F\|_{C^m(\mathbb{R}^n)} \leq C\,, \text{ and}$$

(29) $$|F(x) - f(x)| \leq C \cdot \sigma(x) \text{ on } E\,.$$

From (25),...,(29), and (4), we see that

$$\|F - \tilde{F}\|_{C^m(\mathbb{R}^n)} \leq C\,,$$

$$|(F - \tilde{F})(x)| \leq C \cdot \sigma(x) \text{ on } S^y\,,$$

$$J_y(F - \tilde{F}) = J_y(F) - T^y f\,.$$

Comparing these results with definition (1), we see that

(30) $$J_y(F) - T^y f \in C \cdot \Gamma(y, S^y)\,.$$

Now let $S \subset E$, with $\#(S) \leq k^\#$. By (6) and (30), we have $J_y(F) - T^y f \in C' \cdot \Gamma(y, S)$. This means that there exists $\varphi^S \in C^m(\mathbb{R}^n)$, with

(31) $$J_y(\varphi^S) = J_y(F) - T^y f\,,$$

(32) $$\|\varphi^S\|_{C^m(\mathbb{R}^n)} \leq C'\,,$$

(33) $|\varphi^S(x)| \le C' \cdot \sigma(x)$ on S.

We set $F^S = F - \varphi^S$. Then (31) gives

(34) $J_y(F^S) = T^y f$;

and from (28) and (32) we see that

(35) $\|F^S\|_{C^m(\mathbb{R}^n)} \le C''$.

Also, (29) and (33) yield

(36) $|F^S(x) - f(x)| \le C''' \cdot \sigma(x)$ on S.

Thus, given $S \subset E$ with $\#(S) \le k^\#$, there exists $F^S \in C^m(\mathbb{R}^n)$, satisfying (34), (35), (36). Thus $T^y f \in \mathcal{K}_f(y, k^\#, C'''')$, which is the conclusion of Lemma 2. The proof of the lemma is complete. ■

Lemma 2 above substitutes for Lemma 10.1 in [8]. The proofs of Lemmas 10.2, 10.3 and 10.4 in [8] remain valid here. In place of Lemma 10.5 in [8], we use the following result.

Lemma 3: *Suppose $k^\# \ge (D+1) \cdot k_1^\#$ and $k_1^\# \ge 1$. Let $y \in B(y^0, a_1)$ be given. Then there exists a linear map $T_y^\# : C^m(E,\sigma) \longrightarrow \mathcal{P}$, with the following property*:

If $\|f\|_{C^m(E,\sigma)} \le 1$, then $T_y^\# f \in \mathcal{K}_f^\#(y; k_1^\#, C)$, with C depending only on m and n.

Recall that $\mathcal{K}_f^\#(y; k, M)$ consists of those polynomials $P \in \mathcal{K}_f(y; k, M)$ for which $\partial^\beta P(y) = 0$ for all $\beta \in \mathcal{A}$.

Proof of Lemma 3: We follow the proof of Lemma 10.5 in [8].

By Lemma 10.3 in [8], there exist polynomials $P_\alpha^y \in \mathcal{P}(\alpha \in \mathcal{A})$, satisfying $(\mathbf{WL1})^y$,...,$(\mathbf{WL3})^y$ in [8]. We define

$$T_y^\# f = T^y f - \sum_{\alpha \in \mathcal{A}} (\partial^\alpha (T^y f)(y)) \cdot P_\alpha^y .$$

As promised, $T_y^\#$ is a linear map from $C^m(E,\sigma)$ to $\mathcal{P}$. Moreover, if $\|f\|_{C^m(E,\sigma)} \le 1$, then Lemma 2 above gives $T^y f \in \mathcal{K}_f(y; k^\#, C)$. Therefore, the proof of Lemma 10.5 in [8] applies, with $P = T^y f$ and $\tilde{P} = T_y^\# f$. (The constant 2 in (10.27) in [8] has to be replaced by C, but that has no effect on the rest of the argument.) Thus, as in the proof of Lemma 10.5 in [8], we find that $\tilde{P} \in \mathcal{K}_f^{\#}(y; k_1^{\#}, C'')$. This is the desired property of $T_y^{\#} f$. The proof of Lemma 3 is complete. ■

11. Good News

Again, we place ourselves in the setting of Section 9, and we assume (**SU0,...,5**).

We define the cube Q° and its Calderón-Zygmund decomposition as in Section 11 of [8]. The good news is that all the arguments in Sections 11, ..., 14 of [8] work here as well. In particular, we have the crucial Lemma 14.3 from [8], which we restate here in a slightly weaker form than in [8].

Lemma GN: *Let $y \in Q^{**}$ and $y' \in (Q')^{**}$, where Q and Q' are CZ cubes. Let $f \in C^m(E, \sigma)$, and let $P \in \mathcal{K}_f^{\#}(y; k_A^{\#}, C)$ and $P' \in \mathcal{K}_f^{\#}(y'; k_A^{\#}, C)$ be given, where C depends only on m, n; and assume*

(1) $k^{\#} \geq (D+1) \cdot k_A^{\#}$ and $k_A^{\#} \geq (D+1)^2 \cdot k_{\text{old}}^{\#}$.

If the cubes Q and Q' abut, then we have

(2) $|\partial^{\beta}(P' - P)(y')| \leq C' \cdot (a_1)^{-(m+1)} \, \delta_Q^{m-|\beta|}$ for all $\beta \in \mathcal{M}$.

Here, C' depends only on m and n.

12. Proof of Lemmas SU.I and PP2

In this section, we prove Lemma SU.I. By Lemma SU.II, this will prove Lemma PP2 as well. We place ourselves in the setting of Section 9, and assume (**SU0,...,5**). In particular,

(1) E is a given, finite subset of $\mathbb{R}^n$,

(2) $\sigma : E \longrightarrow (0, \infty)$ is given, and

(3) $\mathcal{A} \subset \mathcal{M}$ is given.

We use the Calderón-Zygmund decomposition from Section 11 of [8]. Let $Q_\nu(1 \le \nu \le \nu_{\max})$ be the CZ cubes, and let $\delta_\nu = \delta_{Q_\nu}$= diameter of Q_ν, y_ν= center of Q_ν. Recall that

(4) $\delta_\nu \le a_1 \le 1$ for each ν,

thanks to (11.3) in [8].

We take

(5) $k^{\#} = (D+1)^3 \cdot k^{\#}_{\text{old}}$.

Let $\tilde{\theta}_\nu(1 \le \nu \le \nu_{\max})$ be a cut-off function, with the following properties.

(6) $0 \le \tilde{\theta}_\nu \le 1$ on $\mathbb{R}^n$, $\tilde{\theta}_\nu = 1$ on Q^*_ν, $\operatorname{supp} \tilde{\theta}_\nu \subset Q^{**}_\nu$,

(7) $|\partial^\beta \tilde{\theta}_\nu| \le C \cdot \delta_\nu^{-|\beta|}$ for $\beta \in \mathcal{M}$.

Throughout this section, we write c, C, C', etc. to denote constants determined by m and n.

Fix $\nu(1 \le \nu \le \nu_{\max})$.

Recall that, since Q_ν is a CZ cube, it is OK. (See section 11 of [8] for the notion of an OK cube.) Thus,

(8) For each $y \in Q^{**}_\nu$, we are given $\bar{\mathcal{A}}^y < \mathcal{A}$, and polynomials $\bar{P}^y_\alpha \in \mathcal{P}(\alpha \in \bar{\mathcal{A}}^y)$ satisfying (**OK1**), (**OK2**), (**OK3**) in [8].

The following result is straightforward.

Lemma 1: *The hypotheses of Lemma CMIA in Section 8 hold here, with* $A = (a_1)^{-(m+1)}$, *for the set* E, *the function* σ, *the cube* Q_ν, *the sets of multi-indices* $\mathcal{A}$ *and* $\bar{\mathcal{A}}^y(y \in Q^{**}_\nu)$, *and the polynomials* $\bar{P}^y_\alpha(y \in Q^{**}_\nu, \alpha \in \bar{\mathcal{A}}^y)$.

Proof: The hypotheses of Lemma CMIA are as follows:

- The STRONG MAIN LEMMA holds for all $\bar{\mathcal{A}} < \mathcal{A}$. (That's just (**SU1**), which we are assuming here.)
- $E \subset \mathbb{R}^n$ is finite, and $\sigma : E \longrightarrow (0, \infty)$. (That's contained in (1) and (2).)

- For each $y \in Q_\nu^{**}$, we are given $\bar{\mathcal{A}}^y < \mathcal{A}$ and $\bar{P}_\alpha^y (\alpha \in \bar{\mathcal{A}}^y)$. (That's immediate from (8).)
- Conditions (**G1**), (**G2**), (**G3**) hold, with $A = (a_1)^{-(m+1)}$. (That's immediate from (**OK1**), (**OK2**), (**OK3**) for Q_ν; these conditions hold, thanks to (8).)

The proof of Lemma 1 is complete. ■

From Lemma 1 and Lemma CMIA, we obtain a linear operator

(9) $\mathcal{E}_\nu : C^m(E,\sigma;\delta_\nu) \longrightarrow C^m(\mathbb{R}^n;\delta_\nu)$, satisfying

(9a) $\|\mathcal{E}_\nu\| \leq A'$, and

$$(10)\quad \left[\begin{array}{l} |\mathcal{E}_\nu f(x) - f(x)| \;\leq\; A' \cdot \|f\|_{C^m(E,\sigma;\delta_\nu)} \cdot \delta_\nu^{-m} \cdot \sigma(x) \\ \\ \text{for all } f \in C^m(E,\sigma;\delta_\nu) \text{ and all } x \in E \cap Q_\nu^* \,. \end{array}\right]$$

Here, and throughout this section, A', A'', A, a, etc., denote constants determined by a_1, m, n.

Next, we bring in Lemma 3 from Section 10, applied with $k_1^\# = (D+1)^2 \cdot k_{\text{old}}^\#$, $y = y_\nu$. (Note that $y_\nu \in B(y^0, a_1)$ as required in Lemma 3, since $y_\nu \in Q_\nu \subseteq Q^\circ \subset B(y^0, a_1)$.) Thus, we obtain a linear map

(11) $T_\nu^\# : C^m(E,\sigma) \longrightarrow \mathcal{P}\,,$

satisfying the following:

(12) If $\|f\|_{C^m(E,\sigma)} \leq 1,$ then $T_\nu^\# f \in \mathcal{K}_f^\#(y_\nu; (D+1)^2 \cdot k_{\text{old}}^\#, C)\,.$

Using the map $T_\nu^\#$ and the cut-off function $\tilde{\theta}_\nu$ (see (6), (7)), we define a linear map

(13) $L_\nu : C^m(E,\sigma) \longrightarrow C^m(E,\sigma;\delta_\nu)$

by setting

(14) $(L_\nu f)(x) = (f(x) - (T_\nu^\# f)(x)) \cdot \tilde{\theta}_\nu(x)$ for all $x \in \mathbb{R}^n\,.$

<u>**Lemma 2**</u>: *The norm of L_ν from $C^m(E,\sigma)$ to $C^m(E,\sigma;\delta_\nu)$, is at most $C' \cdot \delta_\nu^m$.*

Proof: Suppose $\|f\|_{C^m(E,\sigma)} \le 1$. Then (12) and the definition of $\mathcal{K}_f^\#$ yield the following.

(15) Let $S \subset E$, with $\#(S) \le k_{\text{old}}^\#$. Then there exists $F^S \in C^m(\mathbb{R}^n)$, with

(16) $\|F^S\|_{C^m(\mathbb{R}^n)} \le C$,

(17) $|F^S(x) - f(x)| \le C \cdot \sigma(x)$ on S, and

(18) $J_{y_\nu}(F^S) = T_\nu^\# f$.

From (16), (18), and Taylor's theorem, we have

$$|\partial^\beta(F^S - T_\nu^\# f)| \le C\delta_\nu^{m-|\beta|} \text{ on } Q_\nu^{**}, \text{ for } |\beta| \le m.$$

Together with properties (6), (7) of $\tilde{\theta}_\nu$, this implies that

(19) $|\partial^\beta\{\tilde{\theta}_\nu \cdot (F^S - T_\nu^\# f)\}| \le C\delta_\nu^{m-|\beta|}$ on $\mathbb{R}^n$, for $|\beta| \le m$.

On the other hand, (6), (14) and (17) show that

$|\{\tilde{\theta}_\nu \cdot (F^S - T_\nu^\# f)\}(x) - L_\nu f(x)| = |\tilde{\theta}_\nu(x) \cdot (F^S(x) - f(x))|$

$\le |F^S(x) - f(x)| \le C \cdot \sigma(x)$ for all $x \in S$. Together with (19), this shows the following:

(20) Given $S \subset E$ with $\#(S) \le k_{\text{old}}^\#$, there exists $\tilde{F}^S \in C^m(\mathbb{R}^n)$, with

$\|\partial^\beta \tilde{F}\|_{C^0(\mathbb{R}^n)} \le C\delta_\nu^{m-|\beta|}$ for $|\beta| \le m$ and

$|\tilde{F}^S(x) - L_\nu f(x)| \le C \cdot \sigma(x)$ for all $x \in S$.

Comparing (20) with the definition of the $C^m(E,\sigma;\delta)$ norm, we learn that

(21) $\|L_\nu f\|_{C^m(S,\sigma|_S;\delta_\nu)} \le C \cdot \delta_\nu^m$ for all $S \subset E$ with $\#(S) \le k_{\text{old}}^\#$.

We recall from Lemma CMIA that $k_{\text{old}}^\# \ge k_{sw}^\#(m,n)$. Consequently, (21) and the Corollary in section 2 together imply that

$$\|L_\nu f\|_{C^m(E,\sigma;\delta_\nu)} \le C \cdot \delta_\nu^m.$$

This holds whenever $\|f\|_{C^m(E,\sigma)} \leq 1$.

This proof of Lemma 2 is complete. ∎

Next, we introduce a partition of unity on Q^0. We no longer fix ν. For each $\nu (1 \leq \nu \leq \nu_{\max})$, we introduce a cut-off function $\hat{\theta}_\nu$, satisfying

(22) $$0 \leq \hat{\theta}_\nu \leq 1 \text{ on } \mathbb{R}^n,\ \hat{\theta}_\nu = 1 \text{ on } Q_\nu, \hat{\theta}_\nu(x) = 0 \text{ for } \mathrm{dist}(x, Q_\nu) > \hat{c}\delta_\nu\,,$$

and

(23) $$|\partial^\beta \hat{\theta}_\nu| \leq C\delta_\nu^{-|\beta|} \text{ for } |\beta| \leq m\,.$$

Taking $\hat{c}$ small enough in (22), and recalling Lemma 11.2 in [8], we obtain the following.

(24) If Q_μ contains a point of $\mathrm{supp}\,\hat{\theta}_\nu$, then Q_μ and Q_ν coincide or abut.

Define $\theta_\nu = \hat{\theta}_\nu \Big/ \left(\sum_\mu \hat{\theta}_\mu\right)$ on Q^0. From (22) , (23), (24), the Corollary to Lemma 11.1 in [8], and Lemma 11.2 in [8], we obtain:

(25) $$\sum_{1 \leq \nu \leq \nu_{\max}} \theta_\nu = 1 \text{ on } Q^0\,.$$

(26) $0 \leq \theta_\nu \leq 1$ on Q°.

(27) $|\partial^\beta \theta_\nu| \leq C\delta_\nu^{-|\beta|}$ for $|\beta| \leq m$.

(28) $\theta_\nu = 0$ outside Q_ν^*.

(29) If $x \in Q_\mu$, then $\theta_\nu = 0$ in a neighborhood of x, unless Q_μ and Q_ν coincide or abut.

Now we define

(30) $$\tilde{\mathcal{E}}f = \sum_{1 \leq \nu \leq \nu_{\max}} \theta_\nu \cdot [T_\nu^\# f + \mathcal{E}_\nu(L_\nu f)] \text{ on } Q^\circ\,.$$

Note that θ_ν and $\tilde{\mathcal{E}}f$ are defined only on Q°. Since $T_\nu^\#$, $\mathcal{E}_\nu$, and L_ν are linear, (30) shows that

(31) $\tilde{\mathcal{E}}$ is a linear map from $C^m(E,\sigma)$ to $C^m(Q^\circ)$.

Suppose that f is given with

(32) $\|f\|_{C^m(E,\sigma)} \leq 1\,.$

Then, for each ν, we have

(33) $\|L_\nu f\|_{C^m(E,\sigma;\delta_\nu)} \leq C\delta_\nu^m$

by Lemma 2. Hence, (9), (9a), (10) yield the estimates

(34) $\|\mathcal{E}_\nu(L_\nu f)\|_{C^m(\mathbb{R}^n;\delta_\nu)} \leq A'' \cdot \delta_\nu^m\,,$

(35) $|\mathcal{E}_\nu(L_\nu f)(x) - L_\nu f(x)| \leq A'' \cdot \sigma(x)$ for all $x \in E \cap Q_\nu^*\,.$

For all $x \in E \cap Q^\circ$, we have

$$|\theta_\nu \cdot [T_\nu^\# f + \mathcal{E}_\nu(L_\nu f)](x) - \theta_\nu \cdot f(x)| =$$

$$\theta_\nu(x) \cdot |T_\nu^\# f(x) + [\mathcal{E}_\nu(L_\nu f)(x) - L_\nu f(x)] + L_\nu f(x) - f(x)| =$$

$$\theta_\nu(x) \cdot |T_\nu^\# f(x) + [\mathcal{E}_\nu(L_\nu f)(x) - L_\nu f(x)] + \tilde{\theta}_\nu(x) \cdot [f(x) - T_\nu^\# f(x)] - f(x)|$$

$$= \theta_\nu(x) \cdot |T_\nu^\# f(x) + [\mathcal{E}_\nu(L_\nu f)(x) - L_\nu f(x)] + [f(x) - T_\nu^\# f(x)] - f(x)|$$

(because $\tilde{\theta}_\nu = 1$ on supp θ_ν; see (6) and (28))

$$= \theta_\nu(x) \cdot |\mathcal{E}_\nu(L_\nu f)(x) - L_\nu f(x)| \leq \theta_\nu(x) \cdot A'' \cdot \sigma(x)$$

(thanks to (35) when $x \in Q_\nu^*$, and thanks to (28) when $x \notin Q_\nu^*$). Summing over ν, and recalling (25) and (30), we find that

(36) $|\tilde{\mathcal{E}} f(x) - f(x)| \leq A'' \cdot \sigma(x)$ for all $x \in E \cap Q^\circ\,.$

We prepare to estimate the derivatives of $\tilde{\mathcal{E}} f$. To do so, we note that (12) and (32) yield

$$|\partial^\beta (T_\nu^\# f)\,(y_\nu)| \leq C \text{ for } |\beta| \leq m-1\,,$$

since we may take S=empty set in the definition of $\mathcal{K}_f^\#(y_\nu; (D+1)^2 \cdot k_{\text{old}}^\#, C)$. For $|\beta| = m$, we have $\partial^\beta(T_\nu^\# f) \equiv 0$, since $T_\nu^\# f \in \mathcal{P}$. Thus,

(37) $|\partial^\beta (T_\nu^\# f)(y_\nu)| \leq C$ for $|\beta| \leq m\,.$

Since $T_\nu^\# f \in \mathcal{P}$, (37) and (4) show that

$$(38) \qquad |\partial^\beta (T_\nu^\# f)| \leq C \text{ on } Q_\nu^*, \text{ for } |\beta| \leq m\,.$$

We need to compare $T_\nu^\# f$ with $T_\mu^\# f$ when Q_ν and Q_μ abut.

From (12) and (32), we have

$T_\nu^\# f \in \mathcal{K}_f^\#(y_\nu; (D+1)^2 \cdot k_{\text{old}}^\#, C)$, and

$T_\mu^\# f \in \mathcal{K}_f^\#(y_\mu; (D+1)^2 \cdot k_{\text{old}}^\#, C)$.

Hence, we may apply Lemma GN (in section 11), with $k_A^\# = (D+1)^2 \cdot k_{\text{old}}^\#$. (See (5).) Thus,

$$|\partial^\beta (T_\mu^\# f - T_\nu^\# f)(y_\mu)| \leq A \cdot \delta_\nu^{m-|\beta|} \text{ for } |\beta| \leq m-1\,.$$

Recalling Lemma 11.2 in [8], and recalling that $T_\mu^\# f, T_\nu^\# f \in \mathcal{P}$, we conclude that

$$(39) \qquad \left[\begin{array}{l} |\partial^\beta (T_\mu^\# f - T_\nu^\# f)| \leq A' \cdot \delta_\mu^{m-|\beta|} \text{ on } Q_\nu^* \cup Q_\mu^* \text{ for } |\beta| \leq m\,, \\ \\ \text{whenever } Q_\mu \text{ and } Q_\nu \text{ abut.} \end{array}\right]$$

We are almost ready to estimate the derivatives of $\tilde{\mathcal{E}} f$. It is convenient to set

$$P_\nu = T_\nu^\# f,\ F_\nu = \mathcal{E}_\nu(L_\nu f)\,,\ \tilde{F} = \tilde{\mathcal{E}} f\,.$$

Thus,

$P_\nu \in \mathcal{P}$, $F_\nu \in C^m(\mathbb{R}^n)$, and we have the following estimates.

- $|\partial^\beta P_\nu| \leq C$ on Q_ν^*, for $|\beta| \leq m$. (See (38).)
- $|\partial^\beta (P_\nu - P_\mu)| \leq A' \cdot \delta_\mu^{m-|\beta|}$ on $Q_\mu^* \cup Q_\nu^*$ for $|\beta| \leq m$, whenever Q_μ and Q_ν abut. (See (39).)
- $|\partial^\beta F_\nu| \leq A'' \cdot \delta_\nu^{m-|\beta|}$ on $\mathbb{R}^n$, for $|\beta| \leq m$ (See (34).)

  Moreover,
- $\tilde{F} = \sum_{1 \leq \nu \leq \nu_{\max}} \theta_\nu \cdot [P_\nu + F_\nu]$ (See (30).)

Thanks to the above bullets and properties (25),...,(29) of the θ_ν's, the discussion in Section 15 of [8], starting at (15.32) and ending at (15.40) there, applies here as well. (The idea goes back to Whitney.) In particular, estimate (15.40) there yields here the estimate

$$(40) \qquad |\partial^\beta(\tilde{\mathcal{E}}f)(x)| \leq A''' \text{ for all } x \in Q^\circ,\ |\beta| \leq m\,.$$

The extension operator has the good property that (36) and (40) hold whenever f satisfies (32). However, $\tilde{\mathcal{E}}f$ is only defined on Q°.

To remedy this, we pick a cut-off function $\theta^\circ \in C^m(\mathbb{R}^n)$, with

$$\theta^\circ = 1 \text{ on } B(y^0, c'a_1)\,,\ \operatorname{supp}\theta^\circ \subset Q^\circ\,, 0 \leq \theta^\circ \leq 1 \text{ on } \mathbb{R}^n\,,\ \text{and}$$

$$|\partial^\beta\theta^\circ| \leq Ca_1^{-|\beta|} \text{ for } |\beta| \leq m\,.$$

Setting $\mathcal{E}f = \theta \cdot (\tilde{\mathcal{E}}f)$, we obtain a linear operator

$$(41) \qquad \mathcal{E} : C^m(E,\sigma) \longrightarrow C^m(\mathbb{R}^n)\,.$$

From (40) and the defining properties of θ°, we see that

$$(42) \qquad \|\mathcal{E}f\|_{C^m(\mathbb{R}^n)} \leq A_4 \text{ if } \|f\|_{C^m(E,\sigma)} \leq 1\,.$$

From (36) and the defining properties of θ°, we see that

$$(43) \quad |\mathcal{E}f(x) - f(x)| \leq A'' \cdot \sigma(x) \text{ for all } x \in E \cap B(y^0, c'a_1)\,,\ \text{provided } \|f\|_{C^m(E,\sigma)} \leq 1\,.$$

The conclusions, (a) and (b), of Lemma SU.I, are immediate from (42) and (43). (We may take $a = c'a_1$.) The proofs of Lemmas SU.I and PP2 are complete. ■

13. PROOF OF LEMMA PP3

In this section, we prove Lemma PP3. We fix $\mathcal{A} \subset \mathcal{M}$, and assume that the WEAK MAIN LEMMA holds for all $\bar{\mathcal{A}} \leq \mathcal{A}$. We must show that the STRONG MAIN LEMMA holds for $\mathcal{A}$. We may assume that the WEAK MAIN LEMMA holds for all $\bar{\mathcal{A}} \leq \mathcal{A}$, with $k^\#$ and a_0 independent of $\bar{\mathcal{A}}$. (Although each $\bar{\mathcal{A}} \leq \mathcal{A}$ gives rise to its own $k^\#$ and a_0, we may simply use the maximum of all the $k^\#$, and the minimum of all the a_0, arising in the WEAK MAIN LEMMA for all $\bar{\mathcal{A}} \leq \mathcal{A}$.) Fix $k^\#$ and a_0 as in the WEAK MAIN LEMMA for $\bar{\mathcal{A}} \leq \mathcal{A}$.

Let E, σ, y^0, $P_\alpha (\alpha \in \mathcal{A})$ satisfy the hypotheses of the STRONG MAIN LEMMA for $\mathcal{A}$. Without loss of generality, we may suppose

(1) $y^0 = 0$.

We want to find a linear operator $\mathcal{E} : C^m(E, \sigma) \longrightarrow C^m(\mathbb{R}^n)$, satisfying (**SL4,5**).

In this section, we say that a constant is "controlled" if it is determined by C, m, n in the hypotheses (**SL1,2,3**) of the STRONG MAIN LEMMA for $\mathcal{A}$. We write c, C, C', C_1, etc. to denote controlled constants. Also, we introduce a small constant $\bar{a}$ to be picked later. Initially, we do not assume that $\bar{a}$ is a controlled constant. We say that a constant is "weakly controlled" if it is determined by $\bar{a}$, together with C, m, n in (**SL1,2,3**). We write $c(\bar{a})$, $C(\bar{a})$, $C'(\bar{a})$, etc., to denote weakly controlled constants. Note that the constants $k^\#$ and a_0 are controlled. We assume that

(2) $\bar{a}$ is less than a small enough controlled constant.

We proceed as in Sections 16 and 17 of [8]. Section 16 of [8] goes through unchanged here. We introduce the linear map

(3) $T : (\hat{x}_1, \ldots, \hat{x}_n) \mapsto (\lambda_1 \hat{x}_1, \ldots, \lambda_n \hat{x}_n)$, with $\lambda_1, \ldots, \lambda_n > 0$ picked as in Section 17 of [8].

We define

(4) $\hat{E} = T^{-1}(E)$, $\hat{\sigma} = \sigma \circ T$.

As in Section 17 of [8], we may construct a set of multi-indices

(5) $\bar{\mathcal{A}} \leq \mathcal{A}$,

for which the following result is valid.

Lemma 1: (**WL1**), (**WL2**), (**WL3**) *hold for the set* $\hat{E}$*, the function* $\hat{\sigma}$*, the set* $\bar{\mathcal{A}}$ *of multi-indices, the base point* $y^0 = 0$*, and for some family of polynomials* $\bar{P}_{\bar{\alpha}}(\bar{\alpha} \in \bar{\mathcal{A}})$*. Moreover, the constant called* C *in hypothesis* (**WL3**) *for* $\bar{\mathcal{A}}$, $\hat{E}$, $\hat{\sigma}$, $y^0 = 0$, $(\bar{P}_{\bar{\alpha}})_{\bar{\alpha}\in\bar{\mathcal{A}}}$*, is weakly controlled.*

To prove Lemma 1, we just repeat the argument from (17.4a) through(17.27), of [8] omitting the discussion of F^S and $\hat{F}^{\hat{S}}$.

Since we are assuming that the WEAK MAIN LEMMA holds for all $\bar{\mathcal{A}} \le \mathcal{A}$, we obtain from (5) and Lemma 1 that there exists a linear operator

(6) $\hat{\mathcal{E}} : C^m(\hat{E}, \hat{\sigma}) \longrightarrow C^m(\mathbb{R}^n)$, satisfying

(7) $\|\hat{\mathcal{E}}\| \le C_1(\bar{a})$, and

(8) $$\left[\begin{array}{l} |\hat{\mathcal{E}}\hat{f}(\hat{x}) - \hat{f}(\hat{x})| \le C_1(\bar{a}) \cdot \|\hat{f}\|_{C^m(\hat{E},\hat{\sigma})} \cdot \hat{\sigma}(\hat{x}) \\ \\ \text{for all } \hat{f} \in C^m(\hat{E},\hat{\sigma}), \text{ and for all } \hat{x} \in \hat{E} \cap B(0, c_1(\bar{a})) \end{array}\right]$$

Now, given $f \in C^m(E, \sigma)$, we define $\hat{f} = f \circ T$ on $\hat{E}$, then set

(9) $\mathcal{E}f = (\hat{\mathcal{E}}\hat{f}) \circ T^{-1}$.

Thus, $\mathcal{E}$ is a linear operator from $C^m(E, \sigma)$ to $C^m(\mathbb{R}^n)$. Since $\lambda_1, \ldots, \lambda_n$ in (3) satisfy

(10) $c(\bar{a}) \le \lambda_i \le 1$ $(i = 1, \ldots, n)$

(see [8] estimate (17.7)), the operator $f \mapsto \hat{f}$ has norm at most $C(\bar{a})$ as a map from $C^m(E, \sigma)$ to $C^m(\hat{E}, \hat{\sigma})$. Similarly, (9) shows that

$$\|\mathcal{E}f\|_{C^m(\mathbb{R}^n)} \le C(\bar{a}) \cdot \|\hat{\mathcal{E}}\hat{f}\|_{C^m(\mathbb{R}^n)} .$$

Together with (7), this shows that the operator

(11) $\mathcal{E} : C^m(E, \sigma) \longrightarrow C^m(\mathbb{R}^n)$ has norm at most $C_2(\bar{a})$.

Also, from (8), (9), (10), we see that

(12) $|\mathcal{E}f(x) - f(x)| \leq C_1(\bar{a}) \cdot \|\hat{f}\|_{C^m(\hat{E},\hat{\sigma})} \cdot \sigma(x)$ whenever $x \in E$ and $T^{-1}x \in B(0, c_1(\bar{a}))$.

Another application of (10) shows that

(13) $x \in B(0, c_3(\bar{a}))$ implies $T^{-1}x \in B(0, c_1(\bar{a}))$, for a suitable weakly controlled constant $c_3(\bar{a})$.

Again using the fact that the operator $f \mapsto \hat{f}$ has norm at most $C(\bar{a})$ as a map from $C^m(E,\sigma)$ to $C^m(\hat{E},\hat{\sigma})$, we derive from (12) and (13) the following conclusion.

(14) $|\mathcal{E}f(x) - f(x)| \leq C_3(\bar{a}) \cdot \|f\|_{C^m(E,\sigma)} \cdot \sigma(x)$, whenever $f \in C^m(E,\sigma)$ and $x \in E \cap B(0, c_3(\bar{a}))$.

Thus, if $\bar{a}$ satisfies (2), then the operator $\mathcal{E}$ satisfies (11) and (14). We now take $\bar{a}$ to be a controlled constant, small enough to satisfy (2). Then the constants $C_2(\bar{a})$, $C_3(\bar{a})$, $c_3(\bar{a})$ are determined entirely by C, m, n in hypotheses (**SL1,2,3**). Hence, (11) and (14) are the desired properties (**SL4,5**) for the linear operator $\mathcal{E}$. Thus, the STRONG MAIN LEMMA holds for $\bar{\mathcal{A}}$.

The proof of Lemma PP3 is complete. ∎

14. PROOF OF THEOREM 1

By now, we have proven Lemmas PP1, PP2 and PP3. Consequently, we have established the Local Theorem 1 in Section 5. To pass to Theorem 1, we first prove the following simple result.

Lemma 1: *In the* Local Theorem 1, *the hypothesis* $\sigma : E \longrightarrow [0,\infty)$ *may be relaxed to* $\sigma : E \longrightarrow [0,\infty)$.

Proof: Let $E \subset \mathbb{R}^n$ be finite, let $\sigma : E \longrightarrow [0,\infty)$, and let $y^0 \in \mathbb{R}^n$. Since E is finite there exists a linear operator

(1) $\mathcal{E}^{\text{trivial}} : C^0(E) \longrightarrow C^m(\mathbb{R}^n)$, with

(2) $\mathcal{E}^{\text{trivial}} f(x) = f(x)$ for all $x \in E$.

We have

(3) $\|\mathcal{E}^{\text{trivial}} f\|_{C^m(\mathbb{R}^n)} \leq \Gamma(E) \cdot \|f\|_{C^0(E)}$

for all f, with $\Gamma(E)$ a finite constant depending on E.

We write

(4) $E = E_0 \cup E_1$, with

(5) $E_0 = \{x \in E : \sigma(x) = 0\}$ and $E_1 = \{x \in E : \sigma(x) > 0\}$.

For a small enough $\epsilon > 0$ to be picked below, define

$$\text{(6)} \quad \sigma_\epsilon(x) = \left\{ \begin{array}{lll} \sigma(x) & \text{if} & x \in E_1 \\ \epsilon & \text{if} & x \in E_0 \end{array} \right\}$$

Thus, $\sigma_\epsilon : E \longrightarrow (0, \infty)$, so the Local Theorem 1 applies to E, σ_ϵ. Let $\mathcal{E}$ be the operator provided by the Local Theorem 1 for E, σ_ϵ. Note that

(7) $\|f\|_{C^m(E,\sigma_\epsilon)} \leq \|f\|_{C^m(E,\sigma)}$ for any f, simply because $\sigma_\epsilon \geq \sigma$.

Thus, for any function $f : E \longrightarrow \mathbb{R}$, we have

(8) $\|\mathcal{E} f\|_{C^m(\mathbb{R}^n)} \leq A \|f\|_{C^m(E,\sigma_\epsilon)}$, and

(9) $|\mathcal{E} f(x) - f(x)| \leq A \|f\|_{C^m(E,\sigma_\epsilon)} \cdot \sigma_\epsilon(x)$

for all $x \in E \cap B(y^0, c')$.

Here A and c' depend only on m and n. From (5), (6), (9), we have

(10) $|\mathcal{E} f(x) - f(x)| \leq A\epsilon \cdot \|f\|_{C^m(E,\sigma_\epsilon)}$

for all f, and for all $x \in E_0 \cap B(y^0, c')$.

We define a linear operator $L : C^m(E, \sigma_\epsilon) \longrightarrow C^0(E)$ by setting

$$(11) \qquad Lf(x) = \left\{ \begin{array}{ll} f(x) - \mathcal{E}f(x) & \text{for } x \in E_0 \cap B(y^0, c') \\ 0 & \text{for all other } x \in E \end{array} \right\}.$$

We then define the linear operator $\tilde{\mathcal{E}} : C^m(E, \sigma_\epsilon) \longrightarrow C^m(\mathbb{R}^n)$, by setting

$$(12) \quad \tilde{\mathcal{E}}f = \mathcal{E}f + \mathcal{E}^{\text{trivial}}(Lf)$$

for all f.

Note that

$$(13) \quad \|\tilde{\mathcal{E}}f\|_{C^m(\mathbb{R}^n)} \leq \|\mathcal{E}f\|_{C^m(\mathbb{R}^n)} + \Gamma(E) \cdot \|Lf\|_{C^0(E)} \text{ (see (3))}$$

$$\leq A\,\|f\|_{C^m(E,\sigma_\epsilon)} + \Gamma(E) \cdot \|Lf\|_{C^0(E)} \text{ (see (8))}$$

$$\leq A\,\|f\|_{C^m(E,\sigma_\epsilon)} + \Gamma(E) \cdot [A\epsilon \cdot \|f\|_{C^m(E,\sigma_\epsilon)}] \text{ (see (10), (11))}$$

$$\leq 2A \cdot \|f\|_{C^m(E,\sigma_\epsilon)},$$

provided we take

$$(14) \quad \epsilon < 1\Big/\Gamma(E).$$

Let $x \in E_1 \cap B(y^0, c')$. Then we have $Lf(x) = 0$ by definition (11), hence $\mathcal{E}^{\text{trivial}}(Lf)(x) = 0$ by (2). Consequently, (9) and (12) show that

$$|\tilde{\mathcal{E}}f(x) - f(x)| = |\mathcal{E}f(x) - f(x)| \leq A\,\|f\|_{C^m(E,\sigma_\epsilon)} \cdot \sigma_\epsilon(x)$$

$= A\,\|f\|_{C^m(E,\sigma_\epsilon)} \cdot \sigma(x)$. (See (6).) Thus,

$$(15) \quad |\tilde{\mathcal{E}}f(x) - f(x)| \leq A\,\|f\|_{C^m(E,\sigma_\epsilon)} \cdot \sigma(x)$$

for all f, and for all $x \in E_1 \cap B(y^0, c')$.

On the other hand, suppose $x \in E_0 \cap B(y^0, c')$. Then (11) gives $Lf(x) = f(x) - \mathcal{E}f(x)$, hence (2) yields $\mathcal{E}^{\text{trivial}}(Lf)(x) = f(x) - \mathcal{E}f(x)$; and therefore (12) implies $\tilde{\mathcal{E}}f(x) = f(x)$. Thus, we have

(16) $|\tilde{\mathcal{E}} f(x) - f(x)| \leq A \|f\|_{C^m(E,\sigma_\epsilon)} \cdot \sigma(x)$ (both sides vanish)

for all f, and for all $x \in E_0 \cap B(y^0, c')$.

From (4), (15), (16), we conclude that

(17) $|\tilde{\mathcal{E}} f(x) - f(x)| \leq A \|f\|_{C^m(E,\sigma_\epsilon)} \cdot \sigma(x)$

for all f, and for all $x \in E \cap B(y^0, c')$.

From (7), (13), (17), we obtain the following:

(18) $\|\tilde{\mathcal{E}} f\|_{C^m(\mathbb{R}^n)} \leq 2A \|f\|_{C^m(E,\sigma)}$

for all f; and

(19) $|\tilde{\mathcal{E}} f(x) - f(x)| \leq A \|f\|_{C^m(E,\sigma)} \cdot \sigma(x)$

for all f, and for all $x \in E \cap B(y^0, c')$.

Since A and c' depend only on m and n, the conclusions of the Local Theorem 1 are immediate from (18) and (19).

The proof of the Lemma is complete. ■

It is now easy to finish the proof of Theorem 1.

Let $E \subset \mathbb{R}^n$ be finite, and let $\sigma : E \longrightarrow [0, \infty)$ be given.

For each $y \in \mathbb{R}^n$, we obtain from Lemma 1 above a linear operator

$\mathcal{E}^y : C^m(E, \sigma) \longrightarrow C^m(\mathbb{R}^n)$, satisfying for each $f \in C^m(E, \sigma)$ the estimates

(20) $\|\mathcal{E}^y f\|_{C^m(\mathbb{R}^n)} \leq A \|f\|_{C^m(E,\sigma)}$ and

(21) $|\mathcal{E}^y f(x) - f(x)| \leq A \|f\|_{C^m(E,\sigma)} \cdot \sigma(x)$

for all $x \in B(y, c') \cap E$.

Here A, c' depend only on m and n. For the rest of this section, we use c, C, C', etc., to denote constants depending only on m and n.

We introduce a partition of unity

(22) $\sum_\nu \theta_\nu(x) = 1$ for all $x \in \mathbb{R}^n$, where

(22a) $\|\theta_\nu\|_{C^m(\mathbb{R}^n)} \leq C$,

(23) $0 \leq \theta_\nu \leq 1$ on $\mathbb{R}^n$,

(24) $\operatorname{supp} \theta_\nu \subset B(y_\nu, \frac{1}{2} c')$ with c' as in (21), and

(25) No point of $\mathbb{R}^n$ belongs to more than C of the balls $B(y_\nu, c')$.

For $f \in C^m(E, \sigma)$, we define

(26) $\mathcal{E} f = \sum_\nu \theta_\nu \cdot (\mathcal{E}^{y_\nu} f)$.

We have

(27) $\|\mathcal{E} f\|_{C^m(\mathbb{R}^n)} \leq C \cdot \sup_\nu \| \theta_\nu \cdot (\mathcal{E}^{y_\nu} f) \|_{C^m(\mathbb{R}^n)}$
(by (24), (25), (26))
$\leq C' \sup_\nu \|\mathcal{E}^{y_\nu} f \|_{C^m(\mathbb{R}^n)}$ (see (22a))
$\leq C'' \|f\|_{C^m(E,\sigma)}$ (see (20)).

Also, for $x \in E \cap B(y_\nu, c')$, we have

(28) $|\theta_\nu(x) \mathcal{E}^{y_\nu} f(x) - \theta_\nu(x) f(x)| \leq A \|f\|_{C^m(E,\sigma)} \cdot \theta_\nu(x) \sigma(x)$,

thanks to (21) and (23). On the other hand, for $x \in E \setminus B(y_\nu, c')$, (28) still holds, since both sides are zero, thanks to (24). Thus (28) holds for all $x \in E$. Summing (28) over ν, and recalling (22) and (26), we find that

(29) $|\mathcal{E} f(x) - f(x)| \leq A \|f\|_{C^m(E,\sigma)} \cdot \sigma(x)$ for all $f \in C^m(E, \sigma)$ and all $x \in E$.

Since C'' and A depend only on m and n, estimates (27) and (29) are the conclusions of Theorem 1.

The proof of Theorem 1 is complete. ∎

15. Banach Limits

In this section, we recall the basic properties of Banach limits.

A directed set is a set $\mathcal{D}$ with a partial order $>$, with the property that, given any $E_1, E_2 \in \mathcal{D}$, there exists $E \in \mathcal{D}$, with $E \geq E_1$ and $E \geq E_2$.

Let $\mathcal{D}$ be a directed set. A $\mathcal{D}$-sequence is a function from $\mathcal{D}$ to the real numbers. We denote $\mathcal{D}$ sequences by $\vec{\xi} = (\xi_E)_{E\in\mathcal{D}}$.

We write $C^\circ(\mathcal{D})$ to denote the vector space of bounded $\mathcal{D}$-sequences, equipped with the sup norm.

From a well-known application of the Hahn-Banach theorem (see, eg. [7]), there exists a linear functional $\ell_{\mathcal{D}} : C^\circ(\mathcal{D}) \longrightarrow \mathbb{R}$, satisfying the estimate

$$\liminf_{E\longrightarrow\infty} \xi_E \leq \ell_{\mathcal{D}}(\vec{\xi}) \leq \limsup_{E\longrightarrow\infty} \xi_E \text{ for all } \vec{\xi} = (\xi_E)_{E\in\mathcal{D}} \in C^\circ(\mathcal{D})\,.$$

Here, $\liminf_{E\longrightarrow\infty} \xi_E = \sup_{\tilde{E}\in\mathcal{D}} (\inf_{E\geq\tilde{E}} \xi_E)$, and $\limsup_{E\longrightarrow\infty} \xi_E = \inf_{\tilde{E}\in\mathcal{D}} (\sup_{E\geq\tilde{E}} \xi_E)$.

The functional $\ell_{\mathcal{D}}$ is far from unique, but we fix some $\ell_{\mathcal{D}}$ as above, and call it a Banach limit.

16. Equivalence of Norms for Finite Sets

In this section, we prove the following straightforward result.

Lemma ENFS: *Let $E \subset \mathbb{R}^n$ be finite, and let $\sigma : E \longrightarrow [0,\infty)$. Then, for each $f : E \longrightarrow \mathbb{R}$, we have*

(1) $c\|f\|_{C^m(E,\sigma)} \leq \|f\|_{C^{m-1,1}(E,\sigma)} \leq C\|f\|_{C^m(E,\sigma)}\,,$

with c and C depending only on m and n.

Proof: The second estimate is immediate from the definitions and the fact that $\|F\|_{C^{m-1,1}(\mathbb{R}^n)} \leq C\|F\|_{C^m(\mathbb{R}^n)}$ for any $F \in C^m(\mathbb{R}^n)$.

Here and throughout this proof, c, C, etc. stand for constants determined by m and n.

To prove the first estimate in (1), we may assume that

(2) $\|f\|_{C^{m-1,1}(E,\sigma)} = 1.$

We must then show that

(3) $\|f\|_{C^m(E,\sigma)} \leq C.$

In view of (2), there exists $F \in C^{m-1,1}(\mathbb{R}^n)$, with

(4) $\|F\|_{C^{m-1,1}(\mathbb{R}^n)} \leq C$, and

(5) $|F(x) - f(x)| \leq C\sigma(x)$ for all $x \in E$.

By convolving F with an approximate identity, we obtain a family of functions $F_\delta \in C^m(\mathbb{R}^n)$, parametrized by $\delta > 0$, with the following properties:

(6) $\|F_\delta\|_{C^m(\mathbb{R}^n)} \leq C\|F\|_{C^{m-1,1}(\mathbb{R}^n)} \leq C'$ (see (4));

(7) $F_\delta \longrightarrow F$ pointwise, as $\delta \longrightarrow 0$.

Let $\epsilon > 0$ be small enough, to be picked later. Since (7) holds and E is finite, we may pick $\delta > 0$ small enough so that we have

(8) $|F_\delta(x) - F(x)| \leq \epsilon$ for all $x \in E$.

From now on, we fix δ satisfying (8) (and depending on ϵ, of course). From (5) and (8), we get

(9) $|F_\delta(x) - f(x)| \leq C\sigma(x) + \epsilon$ for all $x \in E$.

On the other hand, since E is finite, we have the following trivial remark.

(10) Given a function $g \in C^0(E)$, there exists $G \in C^m(\mathbb{R}^n)$, with

(a) $G(x) = g(x)$ for all $x \in E$, and

(b) $\|G\|_{C^m(\mathbb{R}^n)} \leq \Gamma(E) \cdot \|g\|_{C^0(E)}$,

for a finite constant $\Gamma(E)$ depending on E.

In view of (9), there exists a function $g : E \longrightarrow \mathbb{R}$, with

(11) $|g(x)| \leq \epsilon$ for all $x \in E$, and

(12) $|(F_\delta(x) - f(x)) - g(x)| \leq C\sigma(x)$ for all $x \in E$.

Applying (10) to the function g in (11), (12), we obtain a function $G \in C^m(\mathbb{R}^n)$, with the following properties:

(13) $\|G\|_{C^m(\mathbb{R}^n)} \leq \Gamma(E) \cdot \epsilon$; and

(14) $|F_\delta(x) - f(x) - G(x)| \leq C\sigma(x)$ for all $x \in E$.

We pick $\epsilon < 1/\Gamma(E)$, and set $\tilde{F} = F_\delta - G$. From (6) and (13), we see that

(15) $\|\tilde{F}\|_{C^m(\mathbb{R}^n)} \leq C'$.

From (14) we have

(16) $|\tilde{F}(x) - f(x)| \leq C\sigma(x)$ for all $x \in E$.

Estimates (15) and (16) prove (3), thus completing the proof of the Lemma. ■

17. Proof of Theorem 2

We assume here that $m \geq 2$, leaving to the reader the task of modifying our arguments for the case $m = 1$.

Let $E \subset \mathbb{R}^n$, and let $\sigma : E \longrightarrow [0, \infty)$ be given.

Let $\mathcal{D}$ denote the set of all finite subsets $E_1 \subset E$, partially ordered by inclusion: $E_1 \leq E_2$ if and only if $E_1 \subseteq E_2$. Thus $\mathcal{D}$ is a directed set. For each $E_1 \in \mathcal{D}$, we apply Theorem 1, together with Lemma ENFS in Section 16, to obtain a linear operator $\mathcal{E}[E_1] : C^{m-1,1}(E_1, \sigma|_{E_1}) \longrightarrow C^m(\mathbb{R}^n)$, with

(1) $\|\mathcal{E}[E_1]f\|_{C^m(\mathbb{R}^n)} \leq C\|f\|_{C^{m-1,1}(E_1,\sigma|_{E_1})}$

and

(2) $|(\mathcal{E}[E_1]f)(x) - f(x)| \leq C\sigma(x) \cdot \|f\|_{C^{m-1,1}(E_1,\sigma|_{E_1})}$ on E_1, for all f.

Here, and throughout this section, c, C, C', etc., denote constants depending only on m and n.

Note that (1) shows in particular that

(3) $\sup_{E_1 \in \mathcal{D}} |\partial^\beta(\mathcal{E}[E_1]f)(x)| \leq C\|f\|_{C^{m-1,1}(E,\sigma)}$ for all $f \in C^{m-1,1}(E, \sigma)$, $|\beta| \leq m$, $x \in \mathbb{R}^n$.

We define an element $\vec{\xi}(f, \beta, x) \in C^\circ(\mathcal{D})$, by setting

(4) $\vec{\xi}(f, \beta, x) = (\partial^\beta(\mathcal{E}[E_1]f)(x))_{E_1 \in \mathcal{D}}$.

In view of (3), we have $\vec{\xi}(f, \beta, x) \in C^\circ(\mathcal{D})$, and

(5) $\|\vec{\xi}(f, \beta, x)\|_{C^\circ(\mathcal{D})} \leq C\|f\|_{C^{m-1,1}(E,\sigma)}$, for $f \in C^{m-1,1}(E, \sigma)$, $|\beta| \leq m$, $x \in \mathbb{R}^n$.

Applying the Banach limit $\ell_\mathcal{D}$ to $\vec{\xi}(f, \beta, x)$, we obtain functions $F_\beta(x)$, defined by

(6) $F_\beta(x) = \ell_\mathcal{D}(\vec{\xi}(f, \beta, x))$ for $|\beta| \leq m$, $x \in \mathbb{R}^n$, with $f \in C^{m-1,1}(E, \sigma)$ given. Note that the map

(7) $\mathcal{E} : f \longrightarrow F_0$ is linear.

(Here, 0 denotes the zero multi-index.) We will show that $\mathcal{E}$ satisfies the conclusions of Theorem 2.

First, we establish the smoothness of F_0. Immediately from (5), (6) and the properties of the Banach limit, we have

(8) $sup_{x\in\mathbb{R}^n}|F_\beta(x)| \le C\|f\|_{C^{m-1,1}(E,\sigma)}$ for $|\beta| \le m$.

Moreover, for $x, y \in \mathbb{R}^n$, $|\beta| \le m-1$, and $E_1 \in \mathcal{D}$, estimate (1) gives

$$|\partial^\beta(\mathcal{E}[E_1]f)(x) - \partial^\beta(\mathcal{E}[E_1]f)(y)| \le C|x-y| \cdot \|f\|_{C^{m-1,1}(E,\sigma)}\,.$$

Together with (4), this shows that

$$\|\vec{\xi}(f,\beta,x) - \vec{\xi}(f,\beta,y)\|_{C^\circ(\mathcal{D})} \le C|x-y| \cdot \|f\|_{C^{m-1,1}(E,\sigma)}\,.$$

Taking the Banach limit, and recalling (6), we see that

(9) $|F_\beta(x) - F_\beta(y)| \le C|x-y| \cdot \|f\|_{C^{m-1,1}(E,\sigma)}$ for $|\beta| \le m-1$, $x, y \in \mathbb{R}^n$.

Similarly, suppose $x, y \in \mathbb{R}^n (y = (y_1, \ldots, y_n))$, and let $|\beta| \le m-2$. For $j = 1, \ldots, n$, let $\beta[j]$ denote the sum of β and the $j^{\underline{\text{th}}}$ unit multi-index. Then (1) and Taylor's theorem show that

$$|\partial^\beta(\mathcal{E}[E_1]f)(x+y) - \partial^\beta(\mathcal{E}[E_1]f)(x) - \sum_{j=1}^n \left[\partial^{\beta[j]}(\mathcal{E}[E_1]f)(x)\right] y_j| \le$$

$$C|y|^2\,\|f\|_{C^{m-1,1}(E,\sigma)} \qquad \text{for all } E_1 \in \mathcal{D}\,.$$

That is,

$$\|\vec{\xi}(f,\beta,x+y) - \vec{\xi}(f,\beta,x) - \sum_{j=1}^n y_j\vec{\xi}(f,\beta[j],x)\|_{C^\circ(\mathcal{D})} \le$$

$$C|y|^2\,\|f\|_{C^{m-1,1}(E,\sigma)} \qquad \text{(see (4))}\,.$$

Applying the Banach limit and recalling (6), we find that

$$|F_\beta(x+y) - F_\beta(x) - \sum_{j=1}^n y_j F_{\beta[j]}(x)| \le C|y|^2\,\|f\|_{C^{m-1,1}(E,\sigma)}$$

for $x \in \mathbb{R}^n$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, $f \in C^{m-1,1}(E,\sigma)$, $|\beta| \le m-2$.

This shows that

(10) F_β is differentiable, and $\frac{\partial}{\partial x_j} F_\beta = F_{\beta[j]}$, for $|\beta| \le m-2$ and $j = 1, \dots, n$.

From (8), (9), (10), we conclude that $F_0 \in C^{m-1,1}(\mathbb{R}^n)$, and

(11) $\|F_0\|_{C^{m-1,1}(\mathbb{R}^n)} \le C\|f\|_{C^{m-1,1}(E,\sigma)}$.

Thus, we have established the smoothness of F_0.

Next, we estimate $|F_0(x) - f(x)|$ for $x \in E$.

For $\tilde{x} \in E$, and let $\tilde{E}_1 = \{\tilde{x}\} \in \mathcal{D}$. If $E_1 \in \mathcal{D}$ and $E_1 \ge \tilde{E}_1$, then $\tilde{x} \in E_1$, hence (2) implies that

$$|(\mathcal{E}[E_1]f)(\tilde{x}) - f(\tilde{x})| \le C\sigma(\tilde{x}) \cdot \|f\|_{C^{m-1,1}(E,\sigma)} .$$

Consequently, we have

(12) $\limsup_{E_1 \longrightarrow \infty} (\mathcal{E}[E_1]f)(\tilde{x}) \le f(\tilde{x}) + C\sigma(\tilde{x}) \cdot \|f\|_{C^{m-1,1}(E,\sigma)}$, and

(13) $\liminf_{E_1 \longrightarrow \infty} (\mathcal{E}[E_1]f)(x) \ge f(\tilde{x}) - C\sigma(\tilde{x}) \cdot \|f\|_{C^{m-1,1}(E,\sigma)}$.

Also, from (4), (6) with $\beta = 0$, and from the properties of the Banach limit, we have

(14) $\liminf_{E_1 \longrightarrow \infty} (\mathcal{E}[E_1]f)(\tilde{x}) \le F_0(\tilde{x}) \le \limsup_{E_1 \longrightarrow \infty} (\mathcal{E}[E_1]f)(\tilde{x})$.

Inequalities (12), (13), (14) show that

(15) $|F_0(\tilde{x}) - f(\tilde{x})| \le C\sigma(\tilde{x}) \cdot \|f\|_{C^{m-1,1}(E,\sigma)}$.

We have proven (15) for all $\tilde{x} \in E$ and $f \in C^{m-1,1}(E,\sigma)$.

Our estimates (11) and (15) show that the linear operator $\mathcal{E}$ in (7) maps $C^{m-1,1}(E,\sigma)$ to $C^{m-1,1}(\mathbb{R}^n)$, and satisfies the conclusions of Theorem 2. The proof of Theorem 2 is complete.

■

REFERENCES

1. E. Bierstone, P. Milman, W. Pawlucki, *Differentiable functions defined on closed sets. A problem of Whitney*, Inventiones Math. **151, No. 2** (2003), 329-352.

2. Y. Brudnyi, *On an extension theorem*, Funk. Anal. i Prilzhen. **4** (1970), 97-98; English transl. in Func. Anal. Appl. **4** (1970), 252-253.

3. Y. Brudnyi and P. Shvartsman, *A linear extension operator for a space of smooth functions defined on closed subsets of R^n*, Dokl. Akad. Nauk SSSR **280** (1985), 268-270.

4. Y. Brudnyi and P. Shvartsman, *Generalizations of Whitney's extension theorem*, IMRN **3** (1994), 129-139.

5. Y. Brudnyi and P. Shvartsman, *The Whitney problem of existence of a linear extension operator*, J. Geometric Analysis **7, No. 4** (1997), 515-574.

6. Y. Brudnyi and P. Shvartsman, *Whitney's extension problem for multivariate $C^{1,\omega}$ functions*, Trans. Amer. Math. Soc. **353, No. 6** (2001), 2487-2512.

7. N. Dunford and J.T. Schwartz, Linear Operators vol I, pure and appl. math. vol 7, Interscience Publishers, J. Wiley & Sons, New York, 1964.

8. C. Fefferman, *A sharp form of Whitney's extension theorem*, (to appear).

9. G. Glaeser, *Etudes de quelques algebres tayloriennes*, J d'Analyse **6** (1958), 1-124.

10. M. de Guzman, *Differentiation of Integrals in $\mathbb{R}^n$*, lecture notes in Math. vol. 481, Springer Verlag, Berlin-New York, 1975.

11. B. Malgrange, Ideals of Differentiable Functions, Oxford U. Press, 1966.

12. P. Shvartsman, *Lipshitz selections of multivalued mappings and traces of the Zygmund class of functions to an arbitrary compact*, Dokl. Acad. Nauk SSSR **276** (1984), 559-562; English transl. in Soviet Math. Dokl. **29** (1984), 565-568.

13. P. Shvartsman, *On traces of functions of Zygmund classes*, Sibirskyi Mathem. J. **28 N5** (1987), 203-215; English transl. in Siberian Math. J. **28** (1987), 853-863.

14. P. Shvartsman, *Lipschitz selections of set-valued functions and Helly's theorem*, J. Geometric Analysis **12, No. 2** (2002), 289-324.

15. E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton U. Press, 1970.

16. R. Webster, Convexity, Oxford Science Publications, 1994.

17. H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Transactions A.M.S. **36** (1934), 63-89.

18. H. Whitney, *Differentiable functions defined in closed sets I*, Transactions A.M.S. **36** (1934), 369-3897.

19. H. Whitney, *Functions differentiable on the boundaries of regions*, Annals of Math. **35** (1934), 482-485.