On Alexandrov - Fenchel Inequality for k- convex domains

Conference in Honor of Eli Stein

Joint work of Alice Chang & Yi Wang

§1 Background

Minkowski's mixed Volume

A convex, bdd domain $\subseteq \mathbb{R}^{n+1}$

B unit ball in $\mathbb{R}^{n+1}$

$$\mathrm{Vol}(A + tB) = \sum_{k=0}^{n+1} \binom{n+1}{k} \omega_k(A)\, t^k$$

- $\omega_k(A)$: $(n+1)-k$ volume (quar mass integral)

$k=0$ $\qquad \omega_0(A) = $ Volume of A

$k=1$ $\qquad \omega_1(A) = \frac{1}{n+1} |\partial A|$

In general

$$\omega_k(A) = \hat{C}_{n,k} \int_{\partial A} \sigma_{k-1}(\kappa_1, \cdots \kappa_n)\, d\mu$$

$d\mu$ surface measure

$\kappa_1, \cdots \kappa_n$ principal curvature.

$$\hat{C}_{n,k} = \frac{1}{(n+1)\binom{n}{k}}$$

- $\sigma_k(\kappa_1, \cdots \kappa_n)$ k-th elementary symmetric function

$$\sigma_0(\kappa_1, \cdots \kappa_n) \equiv 1$$

$$\sigma_1(\kappa_1, \cdots \kappa_n) = \sum_{i=1}^{n} \kappa_i = H$$ mean curvature

$$\sigma_2(\kappa_1, \cdots \kappa_n) = \sum_{i<j} \kappa_i \kappa_j$$ etc

- $\sigma_k(\kappa_1, \cdots \kappa_n) = \sigma_k(L)$

$L = L_{ij}$ = 2nd fundamental form on ∂A

$\sigma_k = \sigma_k(L)$ = k-th elementary symmetric function of eigenvalues of

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<u>Alexandrov - Fenchel inequality</u> $\Rightarrow$

A convex $\Rightarrow$

$$\left(\frac{\omega_k(A)}{\omega_{n+1}} \right)^{\frac{1}{n+1-k}} \nearrow k$$

$0 \leq k \leq n$

A convex domain in $\mathbb{R}^{n+1}$ $\Rightarrow$ $(*)_k$ holds for all k

$$(*)_k: \quad \left(\int_{\partial A} \sigma_{k-1}(L)\, d\mu_A \right)^{\frac{1}{n-(k-1)}} \leq \bar{C}_{n,k} \left(\int_{\partial A} \sigma_k(L)\, d\mu_A \right)^{\frac{1}{n-k}}$$

where $L = L_{\partial A}$ = 2nd fundamental form for ∂A

$d\mu_A$ surface measure

$\bar{C}_{n,k}$ attained when A = Ball B

$\boxed{\text{Question}}$ Is "convexity" assumption necessary in $(*)_k$?

e.g.

$$(*)_0 \qquad |A|^{\frac{1}{n+1}} \leq \bar{C}_{n,0} \, |\partial A|^{\frac{1}{n}}$$

holds for <u>all</u> bounded domain in $\mathbb{R}^{n+1}$

<u>Definition</u> $\Omega \subseteq \mathbb{R}^{n+1}$ is k-convex

if $\sigma_1(L_{\partial\Omega}), \cdots\cdots \sigma_k(L_{\partial\Omega}) > 0$

denote by $\partial\Omega \in \Gamma_k^+$

Remark: n-convex $\Rightarrow$ convex

Theorem : (P.F. Guan + J. Li , '08)

Ω : k-convex & Star-shaped

then $(*)_m$ holds for all $m \le k$.

Outline of the proof :

$\vec{X}$ position vector of $\partial\Omega$

Consider flow

$$(**) \qquad \frac{\partial \vec{X}}{\partial t} = \frac{\sigma_{k-1}(L)}{\sigma_k(L)} \vec{\nu}$$

(e.g. k=1 is the inverse mean curvature flow

- Gerhard / Urbas proved under Star-shaped assumption, flow $(**)$ exists & converges $t \to \infty$ } Huisken - ~~Illmanen~~ Illmanen
- Guan + Li : monotonicity of the quotient

$$\left(\int_{\partial\Omega} \sigma_{k-1} \right)^{\frac{1}{n+1-k}} \Big/ \left(\int_{\partial\Omega} \sigma_k \right)^{\frac{1}{n-k}}$$

under the flow $(**)$.

PCE

Theorem (Michael - Simon, '73)

$$M^n \overset{i}{\hookrightarrow} \mathbb{R}^{n+k} \quad i: \text{immersion}, \ \exists C_{n,k}, \forall u \geq 0$$

$u \geq 0$

$$\text{MS}): \left(\int_M u^{\frac{n}{n-1}} \underbrace{dv_M}_{\text{from induced metric}} \right)^{\frac{n-1}{n}} \leq C_{n,k} \int_M (|\nabla u| + u|\mathcal{H}|)\, dv_M$$

$\mathcal{H}$ mean-curvature vector of the immersion.

Corollary $\Omega \subseteq \mathbb{R}^{n+1}$

$M = \partial\Omega$, $\mathcal{H} = H\vec{n}$ H mean curv of $\partial\Omega$

$(MS) \Rightarrow (*)_1'$: $H > 0$ then

$$* \quad |\partial\Omega|^{\frac{n-1}{n}} \lesssim \left(\int_{\partial\Omega} H\right)$$

$$(*)_1' : \text{ or } \quad |\partial\Omega|^{\frac{1}{n}} \underset{\uparrow \text{ same constant}}{\lesssim} \left(\int_{\partial\Omega} H\right)^{\frac{1}{n-1}}$$

Remark: $n=2$, Huisken - Illmanen proved

$$(*)_1 : \quad |\partial\Omega|^{\frac{1}{2}} \leq \bar{C} \left(\int_{\partial\Omega} H\right) \quad \underline{\text{when}} \cdots$$

~~when $\partial\Omega$ is "out~~

When $\partial\Omega$ is "outward minimizing"

i.e. $\Omega' \supseteq \Omega \Rightarrow |\partial\Omega'| \geq |\partial\Omega|$

e.g.

is not outward minimizing

In general, it remains an $\boxed{\text{Open}}$ question

what is the best constant in

$$(*)'_1 : \quad |\partial\Omega|^{\frac{1}{n}} \lesssim \Big(\int_{\partial\Omega} H\Big)^{\frac{1}{n-1}}.$$

Denote $\Omega \subseteq \mathbb{R}^{n+1}$.

$$(*)'_k : \Big(\int_{\partial\Omega} \sigma_{k-1}\Big)^{\frac{1}{n-(k-1)}} \lesssim \Big(\int_{\partial\Omega} \sigma_k\Big)^{\frac{1}{n-k}}$$

a dimension constant, depend on n, k.

Outline of Lecture

(1) background

(2) Classical Alexandrov - Fenchel inequality for convex domains
(why it does not lead to $(*)_k$ for k-convex domain)

(3) P. Castillon ('09) proof of M-S inequality using method of "Optimal transport "map"."

(4) Modify (3) to prove Thm of (Y. Wang + C).

§2. Classical Alexandrov – Fenchel inequality

Given $A_1, \cdots A_{n+1}$ convex domain

<u>Generalized</u> Minkowski mixed Volume

$$V(A_1, \cdots A_{n+1})$$

- When $A_1 = \cdots = A_{n+1}$, $V(A, \cdots A) = Vol(A)$
- When $A_1 = \cdots = A_{n+1-k} = A$, $A_j = B$ $j \geq n+1-k$

$$V(\underbrace{A, \cdots A}_{n+1-k}, \underbrace{B, \cdots B}_{k}) = c_{n,k} \int_{\partial A} \sigma_{k-1}(L_{\partial A})$$

eg.

<u>(AF): inequality</u>

$$V^2(A_1, A_2, A_3, \cdots A_{n+1})$$

$$\geq V(A_1, A_1, A_3, \cdots A_{n+1}) \, V(A_2, A_2, A_3, \cdots A_{n+1})$$

$(AF) \Rightarrow (AF)_k$: ~~k≤n~~ $k \leq n-1$

$$\left(\int_{\partial A} \sigma_k(L) \right)^2 \geq \bar{C}_k \left(\int_{\partial A} \sigma_{k+1}(L) \right) \left(\int_{\partial A} \sigma_{k-1}(L) \right)$$

attained when $A = $ Ball

It turns out

$$(AF)_l \quad \forall\, l \geq k$$

A convex $\quad \Downarrow$

$$(*)_k : \left(\int_{\partial A} \sigma_{k-1}\right)^{\frac{1}{n-(k-1)}} \leq \bar{C}\left(\int_{\partial A} \sigma_k\right)^{\frac{1}{n-k}}$$

<u>Example</u>: When $k=1$, under assumption $\Omega \in \Gamma_2^+$, $(AF)_1$ fails.

- $(AF)_1$: $\left(\int_{\partial\Omega} \sigma_0\right)\left(\int_{\partial\Omega} \sigma_2\right) \leq \frac{n-1}{2n}\left(\int_{\partial\Omega} \sigma_1\right)^2$

$2\sigma_2 = H^2 - |L|^2$

i.e. $\exists\ \Omega \in \Gamma_2^+$, $(AF)_1$ fails

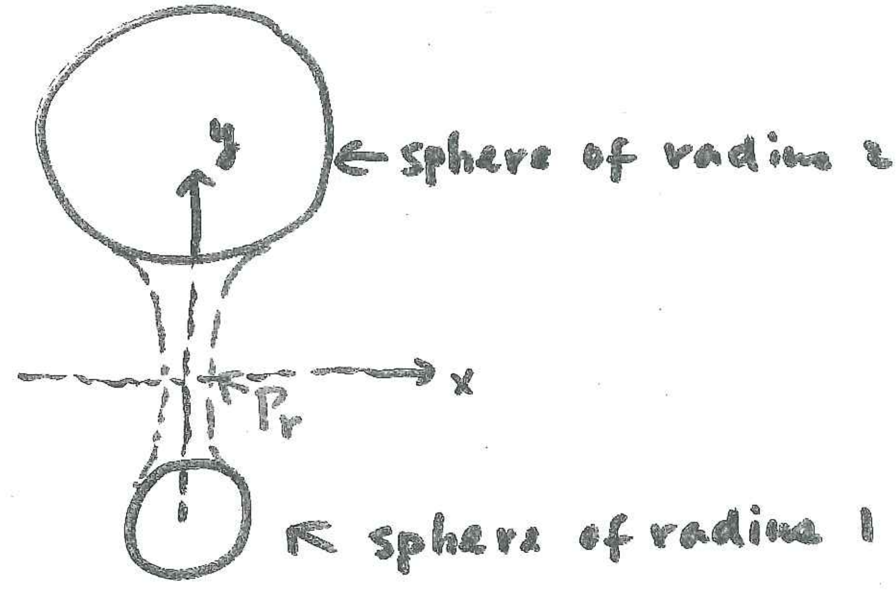
To see this, we rewrite $(AF)_1$ as equivalent to

$$(**) \qquad \int_{\partial\Omega} (H - \bar{H})^2 \leq \frac{n}{n-1} \int_{\partial\Omega} |\mathring{L}|^2$$

$\mathring{L}$ = Traceless part of L

(This part is motivated by some recent work of Topping - De Lellis on a related inequality)

Idea



Choose Γ_r : hypersurface generated by revolving a function f_r along y-axis

- Choose f $\quad \sigma_2(\Gamma_r) = 0$
  $(n \geq 3)$ $\quad \sigma_1(\Gamma_r) > 0$

$$f_r = r f\left(\frac{x}{r}\right)$$

e.g.
$$\begin{cases} f(x) = 2\sqrt{x-1} & n = 3 \quad (x = x_1 \text{ etc}) \\ f(x) = \cosh^{-1} x & n \geq 4 \quad (x = x_1 \text{ etc}) \end{cases}$$

- As $r \to 0$ $\quad \int_{\Gamma_r} |\overset{\circ}{L}|^2 \to 0 \quad (n \geq 3)$

while $|\overset{\circ}{L}| = 0$ on the spheres.

§3. Castillon's re-proof of Michael-Simon inequality

(Special case $M = \partial\Omega$, $\Omega \subseteq \mathbb{R}^{n+1}$)

<u>Facts</u> of Optimal transport map
(Simple form)

D, D^* bdd domain in $\mathbb{R}^n$

$d\mu = F\,dx$, probability measure on D

$d\nu = G\,dy$ " D^*

<u>Problem</u> Find $T: D \to D^*$

$$\inf_{T_\# \mu = \nu} \int_D |x - T(x)|^2 \, d\mu(x)$$

where $T_\# \mu = \nu$ means $\mu(T^{-1}(E)) = \nu(E)$
$\forall\, E \subseteq D^*$

<u>Brenier</u> minimal achieved

$T = \nabla V$ for some V which is <u>convex</u>

$\forall F, G \quad \exists V$ convex

$$\star \quad \boxed{F(x) = G(\nabla V(x)) \operatorname{Det}(\nabla^2 V(x))}$$

- leads to a proof of isoperimetric inequality
- McCann ('94) apply optimal transport method to study Minkowski's inequality for convex body & other inequalities
- Cordero-Nazaret-Villani ('04) apply optimal Transport

$$\|\nabla f\|_{p^*} \le C_n(p) \|f\|_p \qquad p^* = \frac{np}{n-p}$$

get sharp constant $C_n(p)$

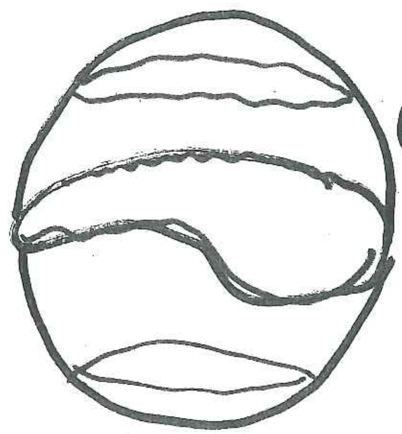
- Figalli + ...

Castillon's Proof of M-S inequality:

$$M^n \xhookrightarrow{i} \mathbb{R}^{n+k} \qquad \forall\, u \geq 0$$

$$\text{(M-S)}: \left(\int_M u^{\frac{n}{n-1}}\, dv_M\right)^{\frac{n-1}{n}} \lesssim \int_M \left(|\nabla u| + |\mathcal{H}|\, u\right) dv_M$$

Special case: $M = \partial\Omega \subseteq \mathbb{R}^{n+1}$, $\mathcal{H} = H\vec{n}$



(M, g)

E n-linear subspace in $\mathbb{R}^{n+k}$ $(k=1)$

g = induced metric surface area

$\downarrow P_E$

$\bar{v} = v \circ P_E$

$\xrightarrow{T = \nabla v}$

$D = P_E(M)$ $\qquad$ $D^* =$ Ball in $\mathbb{R}^n$

$F = (P_E)_\# \mu \dashrightarrow G = \dfrac{\chi_B\, dy}{|B|}$

μ probability measure on M.

$\mu = f\,dx$

Transport equation:

$$(\star) \qquad \omega_n (P_E)_\# f = \det(\nabla^2 v) \quad \text{on } E$$

Structure equation

$$(S) \qquad D^2 v(x) = \bar{D}^2 \bar{v}\Big|_{T_x M} + \langle \bar{\nabla} \bar{v}, \mathcal{L}_x \rangle$$

$\uparrow$ gradient on M $\qquad \mathcal{L}_x = L \cdot \vec{n}$

Combine $(\star)$ & (S)

$$\Rightarrow \qquad \omega_n (P_E)_\# f \, J_E^2 = \det \bar{D}^2 \bar{v}\, \Big|_{T_x M} \quad \text{on } M$$

$$(P_E)_\# f = \sum_{P_E(x) = \xi} \frac{f(x)}{J_E(x)}$$

$$\Rightarrow \qquad (\omega_n J_E f)^{\frac{1}{n}} \le \left(\det \bar{\nabla}^2 \bar{v}\,\Big|_{TM}\right)^{\frac{1}{n}}$$

$(\star)'$

$$\le \frac{\bar{\Delta} \bar{v}}{n}$$

$$= \frac{\Delta v}{n} - \langle \bar{\nabla} \bar{v}, H \vec{n} \rangle$$

Choose $\quad f = \dfrac{u^{\frac{n}{n-1}}}{\int_M u^{\frac{n}{n-1}}}$ $\quad$ mutiply $(\star)'$ by u and integrate

$$\Big(\int_{\substack{M\\ \shortparallel \\ \partial\Omega}} J_E^{\frac{1}{n}}\, u^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}} \lesssim \int_{\partial\Omega} (\Delta v)\, u + \int H u$$

$$= -\int_{\partial\Omega} \nabla v \,\nabla u + \int_{\partial\Omega} H u$$

after justification of distributional Laplace via "Alexandrov sense Laplace"

$$\le \int_{\partial\Omega} |\nabla u| + \int_{\partial\Omega} H u$$

$\because\ |\nabla v| \le 1$

We then basically integrate over all E

use $$\frac{1}{G(n+1,n)} \int_{G(n+1,n)} J_E^{\frac{1}{n}}\, d\mu_E = \alpha_n < \infty$$

get M–S inequality:

$$\alpha_n^{\frac{n-1}{n}} \Big(\int_{\partial\Omega} u^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}} \lesssim \int_{\partial\Omega} |\nabla u| + \int_{\partial\Omega} u H$$

as wanted

Take $M = \partial\Omega$, $u \equiv 1$, we get

$(*)_1'$: $$\Big(\int_{\partial\Omega} 1\Big)^{\frac{1}{n}} \lesssim \Big(\int_{\partial\Omega} H\Big)^{\frac{1}{n-1}}$$

<u>Theorem</u> (<u>Yi Wang</u> + C)

(a): $\Omega \subseteq \mathbb{R}^{n+1}$, $\underline{\Omega \in \Gamma^+_{k+1}}$ $\forall\ 1 \le k \le n$

then $(*)'_m$ holds for all $m \le k$.

(b) (M-S) type Sobolev inequalities:

$\underline{k=2}$, $\underline{\Omega \in \Gamma^+_3}$ then $\forall\ u \geq 0$ smooth

(1) $$\left(\int_{\partial\Omega} u^{\frac{n}{n-2}} d\mu\right)^{\frac{n-2}{n}} \lesssim \int_{\partial\Omega} |D^2 u| + \int_{\partial\Omega} |Du|\, H + \int_{\partial\Omega} u\, \underline{\sigma_2(L)}$$

(2) $$\left(\int_{\partial\Omega} u^{\frac{n-1}{n-2}} \underline{H}\, d\mu\right)^{\frac{n-2}{n-1}} \lesssim \text{same as (1).}$$

$\uparrow$ RHS

For general k, $\underline{\Omega \in \Gamma^+_{k+1}}$, $\forall u \geq 0$

$\forall\ m \le k$, $\forall\ 1 \le j \le m$

$$\left(\int_{\partial\Omega} u^{\frac{n-(m-j)}{n-m}} \underline{\sigma_{m-j}(L)}\right)^{\frac{n-m}{n-(m-j)}}$$

$$\lesssim \sum_{\ell=0}^{m} \left(\int_{\partial\Omega} |D^{m-\ell} u|\, \underline{\sigma_\ell(L)}\, d\mu\right)$$

Proof of Thm (First part)

$M = \partial\Omega$, $\Omega \subseteq \mathbb{R}^{n+1}$, $\boxed{\partial\Omega \in \Gamma_3^+}$

$\Rightarrow$

$$(*)_2' \qquad \left(\int_{\partial\Omega} \sigma_1 \, d\mu\right)^{\frac{1}{n-1}} \lesssim \left(\int_{\partial\Omega} \sigma_2 \, d\mu\right)^{\frac{1}{n-2}}$$

Fix E linear n-space

$\forall$ $f\,dx$ probability measure on M

① $$f J_E \lesssim \det \bar{\nabla}^2 \bar{v}$$

where ∇v: optimal map: $(P_E(M), (P_E)_\# f)$

$$\downarrow$$

$$\left(B, \frac{\chi_B}{|B|}\right)$$

② $$\nabla^2 v = \bar{\nabla}^2 \bar{v}\Big|_{TM} + b(x)\, L$$

$b(x) = -\langle \bar{\nabla} \bar{v}, \vec{n} \rangle \qquad |b| \lesssim 1$

$\bar{\nabla}^2 \bar{v} \geq 0$

To prove $(*)_2'$, we choose $f = \dfrac{\sigma_1}{\int_{\partial\Omega} \sigma_1}$

Take α power of both side of ①

Mutiply by $\sigma_1^{\beta}(a_{ij})$ $\quad \alpha+\beta=1$

Where $\quad a_{ij} = v_{ij} + a\,L_{ij} \qquad a \gg 1$

Using <u>Concavity property</u> of

$$\begin{cases} (\det \bar{\nabla}^2 \bar{v})^{2/n} \leq \sigma_2(\bar{D}^2 v) \\ \left(\dfrac{\sigma_k(A)}{\sigma_l(A)}\right)^{\frac{1}{k-l}} \quad \text{for } A \in \Gamma_k^+,\ l \leq k \end{cases}$$

We reach a position

$$(*)_2'' \quad \left(\int J_E^{\alpha}\sigma_1(L)\right)^{\frac{n-2}{n-1}} \lesssim \int \sigma_2(a_{ij})$$

$$\lesssim \int_{\partial\Omega} \sigma_2(v_{ij}) + \int_{\partial\Omega} \sigma_2(L)$$

<u>Lemma</u> $\quad v \in C^3 \quad |\nabla v| \leq 1$

ⓐ $\displaystyle\int_{\partial\Omega} \sigma_2(v_{ij}) = \int_{\partial\Omega} R_{ij}\, v_i\, v_j$

R_{ij} = Ricci curvature on $\partial\Omega$

ⓑ $\boxed{\partial\Omega \in \Gamma_3^+}$ $\quad \displaystyle\int_{\partial\Omega} R_{ij}\, v_i\, v_j \leq \int_{\partial\Omega} \sigma_2(L)\,|\nabla v|^2 \leq \int_{\partial\Omega} \sigma_2(L)$

P (9)

(a) Follows

$$\nabla_k v_{ij} = \nabla_i v_{kj} + R_{mijk} v_m$$

(b) Follows

↓ Codazzi equation & Bochner's formula

$$R_{mijk} = \underbrace{\cancel{\bar{R}_{mijk}}}_{=0} + L_{mj} L_{ik} - L_{mk} L_{ij}$$

$\Downarrow$

$$R_{ij} = \sigma_2(L) g_{ij} - \frac{\partial \sigma_3(L)}{\partial L_{ij}}$$

To see this: Diagonalize L

$$L = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

$$R_{ij} = \begin{pmatrix} \lambda_1(H-\lambda_1) & & \\ & \ddots & \\ & & \lambda_n(H-\lambda_n) \end{pmatrix}$$

thus for all l

$$\underline{\lambda_l(H-\lambda_l)} + \frac{\partial \sigma_3}{\partial \lambda_l} = \sigma_2(L)$$

$\partial\Omega \in \Gamma_3^+ \Rightarrow \dfrac{\partial \sigma_3}{\partial \lambda_l} > 0 \Rightarrow R_{ij} \le \sigma_2(L) g_{ij}$

General k

~~P(18)~~ P(21)

Basic Lemma $\quad \partial\Omega \in \Gamma_{k+1}^{+}$, $a_{ij} = v_{ij} + a L_{ij}$

$v_{ij} \geq 0$, $|Dv| \leq 1$ $\qquad a >> 1$

$(*)_k''$
$$\int_{\partial\Omega} \sigma_k(a_{ij})\, d\mu_g \lesssim \int_{\partial\Omega} \sigma_k(L_{ij})\, d\mu_g$$

Proof: Use n-linear form of determinant Σ_n

$$\sigma_n(A_{ij}) = \det A_{ij} = \Sigma_n(A_{ij}, \cdots A_{ij})$$

$$\sigma_k(A_{ij}) = \Sigma_n(\underbrace{A, \cdots A}_{k \text{ times}}, \underbrace{I, \cdots I}_{n-k \text{ times}})$$

$(*)$ $\quad k\,\sigma_k(A_{ij}) = A_{ij}\left(T_{k-1}(A)\right)_{ij}$

Newton potential

Key fact (Garding's inequality)

$$A \in \Gamma_k^{+} \Rightarrow T_{k-1}(A) \geq 0$$

We use linear-form $T_{k-1}(A_1, \cdots A_{k-1})$

To illustrate proof for $k = 3$ case.

Proof

$(*)'_3$: $\int_{\partial\Omega} \sigma_3(v_{ij} + a L_{ij}) \lesssim \int_{\partial\Omega} \sigma_3(L_{ij})$

P(21)

1st step: $\int_{\partial\Omega} \Sigma_3(v_{ij}, L_{ij}, L_{ij}) \, d\mu_g = 0$

$$\shortparallel$$

$$v_{ij} \, (T_2(L,L))_{ij}$$

$$\nabla_k L_{ij} = \nabla_j L_{ik} \quad \forall i, j, k$$

$$\Rightarrow \nabla_i (T_2(L,L))_{ij} = 0 \quad \forall ij$$

2nd step

$$\int_{\partial\Omega} \Sigma_3(v_{ij}, v_{ij}, L_{ij})$$

$$= \int_{\partial\Omega} v_{ij} \, (T_2(D^2 v, L))_{ij}$$

$$= -\int_{\partial\Omega} v_i \, D_j (T_2(D^2 v, L))_{ij}$$

$$= \int v_j \, M_{jm} \, v_m \lesssim \int \sigma_3(L)$$

$$M_{jm} = \sigma_2(L) L_{jm} - L^2_{jm} \sigma_1(L) + L^3_{jm}$$

$$= \sigma_3(L) g_{mj} - \frac{\partial \sigma_4}{\partial L_{jm}} \leq \sigma_3(L) g_{mj}$$

$\partial\Omega \in \Gamma_4^+$ $\Longrightarrow$ $\partial\Omega$

P 20

p 22

Step 3

$$\int \Sigma_3 (D^2 v, D^2 v, D^2 v)$$

$$= \int v_j \, N_{jm} \, v_m$$

$$N_{jm} = \Sigma_3 (L, L, D^2 v)\, g_{jm}$$
$$- 3 \left(T_3 (L, L, D^2 v)\right)_{jm}$$
$$- T_2 (L, L)_{km}\, v_{kj}$$

use $v_{ij} = \underset{\geq 0}{\bar{V}_{ij}} + b(x) L_{ij}$ $\quad T_3 (L, L, \bar{V}) \geq 0$

$T_2 (L, L) \bar{V} \geq 0$

We get $\int_{\partial\Omega} \Sigma_3 (D^2 v, D^2 v, D^2 v)$

$$\leq \int \Sigma_3 (L, L, D^2 v) \; |Dv|^2 \quad \longrightarrow = 0$$

$$+ \int \sigma_3 (L) \quad \lesssim \int \sigma_3 (L).$$

General k multi-level induction process

for Proposition

§ Regularity of v

Our estimates holds for $v \in C^3(M)$.

Regularity of Optimal transport map:

In general

$$D \xrightarrow{T = \nabla v} D^* \quad \subseteq \mathbb{R}^n \text{ bdd}$$

$$d\mu = F\,dx \longrightarrow d\nu = G\,dy$$

$$F(x) = G(\nabla v(x)) \det \nabla^2 v(x)$$

v is convex, thus differentiable a.e.

<u>Caffarelli</u>: '92, '92, '96

$0 < \frac{1}{c} \le F, G \le c$, D^* is convex

$$\Rightarrow v \in C^2(D^\circ)$$

<u>In addition</u>: D is concave $\Rightarrow v \in C^2(\bar{D})$

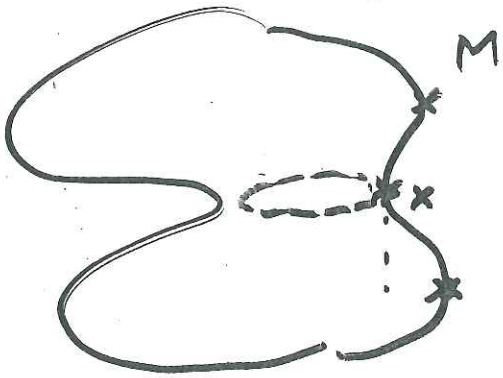
+ additional smoothness of F, G

$$\Rightarrow v \in C^\infty(\bar{D})$$

In our cases: • $D = P_E(\Omega)$ in general is not convex unless Ω is convex

• $F = (P_E)_\# f$ is not bad above at places $J_E = 0$

$\sim \sum \frac{f}{J_E}$



But our estimates

$$\int_{\partial\Omega} \sigma_k(v_{ij} + a L_{ij}) \lesssim \int_{\partial\Omega} \sigma_k(L_{ij})$$

constant depends only on k, a, n

not on C^3-norm of v

Thus we can approximate $D = P_E(\Omega)$ by to Ball, B_ε

enlarge

f by f_ε, F by F_ε, $F_\varepsilon \in C^3$

and let $\varepsilon \to 0$

by $\|F_\varepsilon\|_{C^3} \underset{\varepsilon \to 0}{\to} \infty$

Apply regularity result of Caffarelli, $\varepsilon \to 0$ then

Open Questions:

① Why do we need $\partial\Omega \in \Gamma_{k+1}^{+}$

$(1 < k \leq n-1)$ instead of $\partial\Omega \in \Gamma_{k}^{+}$ to establish $(*)_k$?

② best constant in $(*)_k$ open.