On Eli Stein's square functions

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• Joint work with

Sanghyuk Lee and Keith Rogers.

• Some connections to previous work with Yaryong Heo and Fedya Nazarov.

Classical Littlewood-Paley-Stein function:

$$g(f) = \Big(\int_0^\infty \Big|t\frac{\partial}{\partial t}(P_t * f)\Big|^2 \frac{dt}{t}\Big)^{1/2}$$

where $P_t * f(x)$ is the Poisson integral, or (P_t) a general "nice" approximation of the identity.

Stein expanded Littlewood-Paley theory, using more singular kernels in place of P_t, to make it applicable to interesting geometrical and Fourier analytical questions.

• Example: *Generalized spherical means*

$$\mathcal{A}_t^\beta f(x) = \frac{1}{\Gamma(\beta)} \int_{|y|\le 1} (1-|y|^2)^{\beta-1} f(x-ty)\, dy$$

also defined for $\beta \le 0$ by analytic continuation;

$$\widehat{\mathcal{A}_t^\beta} f(\xi) = C_{\beta,d}\, (t|\xi|)^{\frac{d-2}{2}+\beta} J_{\frac{d-2}{2}+\beta}(t|\xi|)\widehat{f}(\xi)$$

$\beta = 0$: spherical means.

Stein (1976) introduced the "Littlewood-Paley" function

$$\mathcal{G}_\beta f = \Big(\int_0^\infty \Big| t \frac{\partial}{\partial t} \mathcal{A}_t^\beta f \Big|^2 \frac{dt}{t} \Big)^{1/2}$$

to control $\sup_{t>0} |\mathcal{A}_t^{\beta-1/2+\varepsilon} f|$, in particular

$$\sup_t | \int_{S^{d-1}} f(x-ty)d\sigma(y)|.$$

• Sharp L^p bounds for variants of $\mathcal{G}_\beta$ imply bounds for $e^{it\sqrt{-\Delta}}f$ in $L^p(L^2(I))$, with compact time interval I, for initial datum f in L^p-Sobolev spaces.

• There are known implications for classes of radial Fourier multipliers, via

$$m(\sqrt{-\Delta}) = \frac{1}{2\pi} \int \widehat{m}(t) e^{it\sqrt{-\Delta}} dt \qquad \text{and}$$

$$\|m(\sqrt{-\Delta})f\|_p \lesssim \Big(\int |\widehat{m}(t)|^2 (1+|t|^2)^\alpha dt \Big) \times \Big\| \Big(\int \Big| \frac{e^{it\sqrt{-\Delta}} f}{(1+t^2)^{\alpha/2}} \Big|^2 dt \Big)^{1/2} \Big\|_p$$

Bochner-Riesz means $\mathcal{R}_t^\alpha f$ of the Fourier integral

$$\widehat{\mathcal{R}_t^\alpha f} = \Big(1 - \frac{|\xi|^2}{t^2}\Big)_+^\alpha \widehat{f}(\xi).$$

Stein (1958) introduced

$$G^\alpha f = \Big(\int_0^\infty \Big|t\frac{\partial}{\partial t}\mathcal{R}_t^\alpha f\Big|^2 \frac{dt}{t}\Big)^{1/2}$$

to control $\sup_{t>0} |\mathcal{R}_t^{\alpha-1/2+\varepsilon} f|$ for $f \in L^2$ and (and then $f \in L^p$) to prove a.e. convergence for Riesz means of Fourier integrals and series.

Kaneko-Sunouchi (1985):

Uniform pointwise equivalence:

$$G^\alpha f(x) \approx \mathcal{G}_\beta f(x), \quad \beta = \alpha - \frac{d-2}{2}$$

by Plancherel's theorem (for the Mellin transform).

Connections to multipliers

Stein's proof of the Mikhlin-Hörmander multiplier theorem: If $\widehat{Tf} = h\widehat{f}$, $\alpha > d/2$, then there is the *pointwise* inequality

$$g[Tf](x) \le \sup_{t>0} \|\phi h(t\cdot)\|_{L^2_\alpha(\mathbb{R}^d)}\, g^*[f](x)$$

Similar philosophy for radial multipliers (Carbery, Gasper, Trebels, 1984): If $\widehat{T_m f}(\xi) = m(|\xi|)\widehat{f}(\xi)$ then

$$g(T_m f)(x) \le \sup_{t>0} \|\phi m(t\cdot)\|_{L^2_\alpha(\mathbb{R})}\, G^\alpha f(x)$$

based on

$$u(|\xi|) = C(\alpha) \int_{|\xi|}^{\infty} \Big(1 - \frac{|\xi|}{t}\Big)^{\alpha-1} t^\alpha u^{(\alpha)}(t) \frac{dt}{t}.$$

These pointwise inequalities are quite effective.
Q.: Are they effective to even yield sharp endpoint results, $\alpha = d|1/2 - 1/p|$? Here $\alpha > 1/2$.

Discuss

I. L^p inequalities for G_α

II. Weighted norm inequalities

III. Endpoint questions

I. $\mathrm{L^p}$ inequalities

- L^2 inequality for $\alpha > 1/2$, by Plancherel.

- Necessary conditions.

Write

$$t\frac{d}{dt}\mathcal{R}_t^\alpha f = K_t^\alpha * f$$

and, for suitable $\widehat{\eta} \in \mathcal{S}$ vanishing near 0 and $t \sim 1$

$$K_t^\alpha * \eta(x) = e^{it|x|}|tx|^{-\frac{d-1}{2}-\alpha} + \text{better terms}$$

$(|x| \gg 1)$. Thus

$$\Big(\int_1^2 |K_t^\alpha * \eta|^2 dt\Big)^{1/2} \in L^p \text{ iff } \alpha > d(\frac{1}{p} - \frac{1}{2}) + \frac{1}{2}$$

Oscillation does not play any role here.

• Calderón-Zygmund theory and subsequent interpolation gives the necessary and sufficient

$$\|G^\alpha f\|_p \lesssim \|f\|_p, \quad \alpha > d(\frac{1}{p} - \frac{1}{2}) + \frac{1}{2}$$

for $1 \le p < 2$.

For $\alpha = d(\frac{1}{p} - \frac{1}{2}) + \frac{1}{2}$, $1 < p < 2$, there is a weak type (p,p) endpoint result (following Fefferman).

• $\mathbf{p>2}$: More subtle (tied to the Fourier restriction / Bochner-Riesz circle of questions).

• Necessary condition for $p > 2$, $p' < 2$:

$$\Big\| \int_1^2 b(t) K_t^\alpha * \eta \, dt \Big\|_{p'} \lesssim \|b\|_{L^2(\mathbb{R})}$$

i.e.

$$\int_{|x|\ge 1} \Big| \frac{\widehat{b}(|x|)}{|x|^{\frac{d-1}{2}+\alpha}} \Big|^{p'} dx < \infty$$

i.e. $\alpha > d(\frac{1}{p'} - \frac{1}{2}) = d(\frac{1}{2} - \frac{1}{p})$.

Conjecture: For $2 \le p < \infty$,

$$\|G^\alpha f\|_p \lesssim \|f\|_p, \quad \alpha > \max\{d(\frac{1}{2} - \frac{1}{p}), \frac{1}{2}\}$$

Ok for radial functions.

Often a new result on Bochner-Riesz was followed by a new result on G^α:

Carbery (82): $d = 2$.

Partial results by Christ (85) in higher dimensions, also S.(86).

Current result for $d \ge 2$ is:

Thm. [Lee-Rogers-S] For $p > 2 + 4/d$ the conjecture holds.

The p-range corresponds to the range for Tao's bilinear adjoint restriction theorem, i.e. an $L^{p/2}$ bound for $\widehat{g_1 d\sigma_1}\widehat{g_2 d\sigma_2}$, with $d\sigma_1$, $d\sigma_2$ surface measure on 'transversal' portions of the sphere, and $g_1, g_2 \in L^2(S^{d-1})$.

• There are some "endpoint" L^p bounds. One of them takes the form

Thm.: Let $d \geq 2$, $2 + 4/d < p < \infty$, $\gamma > 0$. $\Longrightarrow$

$$\Big\| \Big(\int_{-1}^{1} \Big| e^{it(-\Delta)^{\gamma/2}} f \Big|^2 dt \Big)^{1/2} \Big\|_p \lesssim \|f\|_{L^p_s},$$

$$\frac{s}{\gamma} = d\Big(\frac{1}{2} - \frac{1}{p}\Big) - \frac{1}{2}$$

• L^p_s can be replaced with $B^p_{s,p}$.

• The case $\gamma = 1$ (wave eq.) is most closely related to Stein's square-function but is not exceptional.

• Situation changes for $L^p_s \to L^p(L^r)$ bounds for $r > 2$, then the wave eq. is exceptional.

• The "endpoint" is not an endpoint result for G^α (more about this later).

II. Weighted L^2 norm inequalities

L^p bounds are equivalent with

$$\int [G^\alpha f]^2 w \lesssim \int |f|^2 W$$

for all $w \in L^{(p/2)'}$ with $\|W\|_{(p/2)'} \lesssim \|w\|_{(p/2)'}$

Problems:

• Explicit description of the weight operator $w \mapsto W$.

• Can we choose $W = Ww$ as a maximal operator, in particular is $w \mapsto Ww$ bounded on L^r for $(p/2)' \le r \le \infty$?

• Stein's problem (open even for $d = 2$): Can W be chosen as a variant of a Nikodym maximal function $\mathfrak{N}_q w$, $q < (p/2)'$?

$$\mathfrak{N}_q w := \sup_{\mathfrak{e} \ge 1} \mathfrak{e}^{1-d/q} \sup_{\theta \in S^{d-1}} M_{\theta,\mathfrak{e}} w$$

$\mathfrak{e}$: eccentricity, θ direction.

We want for q near $(p/2)'$:

$$\int [G^\alpha f]^2 w \lesssim \int |f|^2 \mathfrak{W}_q w, \quad \alpha > \frac{d}{2q} \quad (*)$$

with $\mathfrak{W}_q$ bounded on L^r, $q < r \le \infty$.

• $d \ge 2$, $q = 2$: Carbery (85) constructed $\mathfrak{W}_2$ (bounded in $L^r(\mathbb{R}^2)$, $2 < r \le 4$).

• $1 < q \le \frac{d+1}{2}$: Christ (85) observed that

$$\mathfrak{W}_q w = M(|w|^q)^{1/q} \quad \text{works.}$$

• Carbery, S. (2000) constructed $\mathfrak{W}_2$, bounded in $L^r(\mathbb{R}^2)$, $2 \le r \le \infty$.

Thm. [LRS] Let $1 \le q < \frac{d+2}{2}$. There is $\mathfrak{W}_q$, of weak type (q,q), bounded on L^r, $q < r \le \infty$, such that (*) holds for all $\alpha > d/2q$.

Also various endpoint bounds for square functions generated by multipliers $\phi(\delta^{-1}(1 - t|\xi|))$.

Definition of a weight operator

$$W = \left(M \left[M \mathfrak{N}^{negl} w + \sup_{l \in \mathbb{Z}} M W_l P_l w \right]^{1+\varepsilon} \right)^{\frac{1}{1+\varepsilon}}$$

• *Nonessential part*

$$\mathfrak{N}^{negl} w := \sup_{\mathfrak{e} \geq 1} \mathfrak{e}^{-2(\frac{d}{q}-1)} \sup_{\theta \in S^{d-1}} M_{\theta,\mathfrak{e}} w$$

• $P_l w$: dyadic frequency cutoff to frequencies $\approx 2^l$.

• *Essential part*, reminiscent of "grand maximal function". Ignoring $\log \mathfrak{e}$ factors

$$W_l w = \sup_{\mathfrak{e} \geq 2} \mathfrak{e}^{-2(\frac{d}{q}-1)} \sup_{\theta \in S^{d-1}} \left(M_{\theta,\mathfrak{e}} \left[\sup_{\Psi} |\Psi * w|^q \right] \right)^{1/q}$$

where $\sup_\Psi$ is over L^1 normalized Schwartz functions adapted to tubes with eccentricity $\mathfrak{e}$, width 2^{-l}, and direction θ.

• *Significant improvement* in the estimates when W_l acts on functions with cancellation.

Write $W_l w = \sup_{\mathfrak{e}\ge 2} e^{-2(\frac{d}{q}-1)} W_{l,\mathfrak{e}} w$ where

$$W_{l,\mathfrak{e}} w = \Big(\sup_\theta M_{\theta,\mathfrak{e}}\Big[\sup_\Psi |\Psi * g|^q\Big]\Big)^{1/q}$$

(fixed width 2^{-l}, fixed eccentricity, Ψ associated to rectangle in direction θ and parameters $\mathfrak{e}$, 2^{-l}).

Then

$$\big\|W_{l,\mathfrak{e}} w\big\|_{q+\varepsilon} \lesssim \mathfrak{e}^{(d-1)/q}\, \|w\|_{q+\varepsilon}$$

but

$$\big\|W_{l,\mathfrak{e}} P_l w\big\|_{q+\varepsilon} \lesssim \mathfrak{e}^{(d-2)/q}\, \|w\|_{q+\varepsilon}$$

Note that $-2(\frac{d}{q}-1)+\frac{d-2}{q} < 0$ for $q < \frac{d+2}{2}$.

• Reason for the gain: Overlapping properties of dual plates are better in the annulus supp $(\widehat{P_l w})$ than in the ball.

Bilinear analogue of the weighted inequality

Let $\widehat{T_t^\delta f} = \phi(\delta^{-1}(1 - |\xi|^2/t^2))\widehat{f}$.

Let $\widehat{S_1 f}$, $\widehat{S_2 f}$ be supported in two narrow sectors, with transversal directions.

$$\Big| \int \int T_t^\delta S_1 f \, \overline{T_t^\delta S_2 f} \, \frac{dt}{t} \, w(x) dx \Big| \lesssim \delta^{2-d/q} \int g[f]^2 (M[w^q])^{1/q} \, dx \,,$$

with standard Littlewood-Paley square function $g(f)$

Uses Tao's theorem and requires $q < \frac{d+2}{2}$.

III. Endpoint bounds for G^{α}

- There are endpoint (weak type) results on Bochner-Riesz with critical index and certain generalizations (Christ, S., Tao). However endpoint bounds on G^{α} and the corresponding radial multiplier theorems involving L^2_{α} appeared to remain open.

For Bochner-Riesz multipliers there is a natural decomposition into orthogonal pieces (supported on thin annuli). Difficulty with endpoint bounds for G^{α} (or for $m(|\cdot|) \in L^2_{\alpha}$): Dyadic decompositions on the kernel side do not yield almost orthogonal operators on L^2.

Q: What are the endpoint estimates for G^{α} in the range $p > \frac{2d}{d-1}$, $\alpha = d(\frac{1}{2} - \frac{1}{p})$?

Thm. For $d \geq 2$, $p > 2 + 4/(d-1)$,

$$\left\| G^{\alpha}[f] \right\|_{L^p} \lesssim \|f\|_{L^{p2}},$$

for $\alpha = d(\frac{1}{2} - \frac{1}{p})$.

• Implies the sharp inequality

$$\left\| \mathcal{F}^{-1}[m(|\cdot|)\widehat{f}] \right\|_p \lesssim \sup_t \|\phi m(t\cdot)\|_{L^2_{\frac{d}{2}-\frac{d}{p}}} \|f\|_{L^{p2}}$$

• Recall necessary conditions
(after dualization, now $p < 2$).

If $b \in L^2([1,2])$ then

$$\frac{\widehat{b}(|x|)}{(1+|x|)^{\frac{d}{p}-\frac{1}{2}}} \text{ belongs to } L^{p,2}$$

but not necessarily to $L^{p,q}$ for $q < 2$.

• Note: Stein's point of view gives exact end-point bounds.

It is possible to show that (using dualization, atomic decompositions, etc.) it suffices to prove

$$\Big\| \int_1^2 \mathcal{R}_s^{\alpha(p)-1} f(s,\cdot)ds \Big\|_{L^{p,2}} \lesssim \Big\| \Big(\int_1^2 |f(s,\cdot)|^2 ds \Big)^{1/2} \Big\|_p$$

for $1 < p < \frac{2(d+1)}{d+3}$, $\alpha(p) = d(1/p - 1/2)$.

• This should follow "by real interpolation". But what is the object to interpolate?

By Plancherel and explicit formulas for $\mathcal{R}^{\alpha-1}$ the above is deduced from the case $q = 2$ of

$$\Big\| \sum_{j>1} 2^{-jd/p} \int_{2^j}^{2^{j+1}} \eta * \sigma_r * F_j(r,\cdot)dr \Big\|_{L^{pq}} \lesssim \Big\| \{F_j\} \Big\|_{L^p(\ell^q(\mathcal{H}))}$$

here $\widehat{\eta}$ is supported where $|\xi| \approx 1$,
σ_r is surface measure on sphere of radius $r \gg 1$, and $\mathcal{H} = L^2(\frac{dr}{r})$.

• For $p = q$ we can move the weight to the right hand side.

Thm. For $1 \le p < \frac{2(d+1)}{d+3}$

$$\Big\| \sum_{j>1} \int_{2^j}^{2^{j+1}} \eta * \sigma_r * g_j(r,\cdot)dr \Big\|_{L^p} \lesssim \Big(\sum_j 2^{jd} \|g_j\|_{L^p(\mathcal{H})}^p \Big)^{1/p}$$

Let μ_d be the measure "$2^{jd}dx$" on $\mathbb{N} \times \mathbb{R}^d$.

By real interpolation

$$\Big\| \sum_{j>1} \int_{2^j}^{2^{j+1}} \eta * \sigma_r * g_j(r,\cdot)dr \Big\|_{L^{pq}(\mathbb{R}^d)} \lesssim \big\| \vec{g} \big\|_{L^{pq}(\mu_d,\mathcal{H})}$$

and we apply this with $g_j = 2^{-jd/p} F_j$.

For $p \le q \le \infty$ this is followed by

$$\big\| \{2^{-jd/p} F_j\} \big\|_{L^{pq}(2^{jd}dx,\mathcal{H})} \lesssim \big\| \{F_j\} \big\|_{L^p(dx,\ell^q(\mathcal{H}))}$$

which is easy to check for $q = p$ and for $q = \infty$.

Comment on the inequality ($1 \le p < \frac{2(d+1)}{d+3}$).

$$\Big\| \sum_{j>1} \int_{2^j}^{2^{j+1}} \eta * \sigma_r * g_j(r,\cdot)dr \Big\|_{L^p}$$
$$\lesssim \Big(\sum_j 2^{jd} \Big\| \Big(\int_{2^j}^{2^{j+1}} |g_j(r,\cdot)|^2 \frac{dr}{r} \Big)^{1/2} \Big\|_p^p \Big)^{1/p} \qquad (*)$$

(*) is weaker than the inequality

$$\Big\| \sum_{j>1} \int_{2^j}^{2^{j+1}} \eta * \sigma_r * g_j(r,\cdot)dr \Big\|_{L^p}$$
$$\lesssim \Big(\sum_j 2^{jd} \Big\| \Big(\int_{2^j}^{2^{j+1}} |g_j(r,\cdot)|^p \frac{dr}{r} \Big)^{1/p} \Big\|_p^p \Big)^{1/p} \qquad (**)$$

(**) is known only for $1 \le p < \frac{2(d-1)}{d+1}$ (thus no result for $d = 2, 3$). [Heo-Nazarov-S.].

Rewrite as

$$\Big\| \int_1^\infty \eta * \sigma_r * g(r,\cdot)dr \Big\|_{L^p(\mathbb{R}^d)} \lesssim \Big(\int_1^\infty \|g(r,\cdot)\|_p^p \, r^{d-1} dr \Big)^{1/p} \qquad (**)$$

• (**) is an endpoint version of Sogge's L^p wave equation problem (cf. previous work by Wolff). In [H-N-S] it was used to prove a simple characterization of all radial multipliers of $\mathcal{F}L^p$, $1 < p < \frac{2(d-1)}{d+1}$, namely (for $p \le q \le \infty$)

$$\|m(\sqrt{-\Delta})\|_{L^p \to L^{pq}} \approx \sup_{t>0} \|\mathcal{F}^{-1}[\phi m(t|\cdot|)]\|_{L^{pq}} .$$

• Both (*) and (**) may be conjectured for $1 \le p < \frac{2d}{d+1}$.

• There are variable coefficient versions of (**), applicable to FIO's and wave equations on manifolds (joint with Sanghyuk Lee).

• Idea of proof (modifying an idea in [H-N-S])

Prove a "restricted" inequality. Let $E_j \subset \mathbb{Z}^d$ and consider

$$g_j(r,x) = \sum_{z \in E_j} \psi(x-z) b_{j,z}(r)$$

with $|b_{j,z}|_{\mathcal{H}} \leq 1$.

Let

$$A^j = \int_{2^j}^{2^{j+1}} \eta * \sigma_r * g_j(r,\cdot) dr.$$

Need

$$\Big\| \sum_j A_j \Big\|_p^p \lesssim \sum_j 2^{jd} \text{card}(E_j)$$

Decompose $E_j = \cup_{n>0} E_j^n$ so that each E_j^n is a subset of a union of cubes of sidelength 2^j, each containing $\approx 2^n$ points in E_j. Define the corresponding functions g_j^n.

• Let

$$A_j^n = \int_{2^j}^{2^{j+1}} \eta * \sigma_r * g_j^n(r,\cdot) dr.$$

Need

$$\Big\| \sum_j A_j^n \Big\|_p^p \lesssim 2^{-n\varepsilon(p)} \sum_j 2^{jd} \text{card}(E_j)$$

with $\varepsilon(p) > 0$ for $p < \frac{2(d+1)}{d+3}$.

• supp $\sum_j A_j^n$ is contained in a set of size $\lesssim 2^{-n} \sum_j 2^{jd} \text{card}(E_j)$.

• There is an estimate on L^2:

$$\Big\| \sum_j A_j^n \Big\|_2^2 \lesssim n 2^{n\frac{2}{d+1}} \sum_j 2^{jd} \text{card}(E_j)$$

Crucial orthogonality: For $Cn \le k < j - c_1$ the scalar products $\langle A_j^n, A_k^n \rangle$ gain by a factor of $2^{-k(d-1)/2}$ over what is predicted from the estimates for $\|A_j^n\|_2 \|A_k^n\|_2$.