Problems of Harmonic Analysis related to finite type hypersurfaces in 3 space

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joint work with I. Ikromov, and I. Ikromov - M. Kempe

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Introduction: Three inter-related problems

$S =$ smooth, finite type hypersurface in $\mathbb{R}^3$,

$d\mu := \rho d\sigma$, $d\sigma :=$ surface measure on S, $0 \leq \rho \in C_0^\infty(S)$

A

Sharp uniform decay estimates for $\widehat{d\mu}(\xi) := \int_S e^{-i\xi x} d\mu(x)$, $\xi \in \mathbb{R}^3$?

B

$L^p(\mathbb{R}^3)$ - boundedness of the maximal operator $\mathcal{M}f(x) := \sup_{t>0} |A_t f(x)|$, where $A_t f(x) := \int_S f(x - ty) d\mu(y)$.

C

For which p's do we have a Fourier restriction estimate

$$\Big(\int_S |\hat{f}(x)|^2 \, d\mu(x) \Big)^{1/2} \leq C \|f\|_{L^p(\mathbb{R}^3)}, \quad f \in \mathcal{S}(\mathbb{R}^3) \ ?$$

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Short History of these problems

(A) Estimation of oscillatory integrals:

B. Riemann (1854):	appear implicitly in his work
Best understood:	*Convex hypersurfaces of finite line type:*
	B. Randol (1969)
	I. Svensson(1971) H. Schulz (1991)
	J. Bruna, A. Nagel, S. Wainger (1988)
Non-convex case:	
A.N. Varchenko (1976) :	$\int e^{i\lambda\phi(x_1,x_2)} a(x_1,x_2)\,dx$, ϕ analytic
V.N. Karpushkin (1984):	$\int e^{i\lambda(\phi(x_1,x_2)+r(x_1,x_2))} a(x_1,x_2)\,dx$, ϕ analytic

(C) The Fourier-restriction problem: E.M. Stein (1967).
E.M. Stein and P.A. Tomas (1975) :

$$\Big(\int_{S^{n-1}} |\hat{f}(x)|^2\,d\mu(x)\Big)^{1/2} \le C\|f\|_{L^p(\mathbb{R}^n)}$$

iff $p' \ge 2(\frac{2}{n-1}+1)$.

(B) Maximal functions associated to hypersurfaces in $\mathbb{R}^n$:

E.M. Stein (1976)	spheres in $\mathbb{R}^n, n \geq 3$ $(p > n/(n-1))$.
J. Bourgain (1986)	circle in $\mathbb{R}^2$, $(p > 2)$.
A. Iosevich ('94); Marletta('94):	finite type curves
C. Sogge, E.M: Stein ('85):	partial results if $K = 0$ somewhere
Best understood case:	*convex hypersurfaces of finite line type*
Contributions e.g. by:	M. Cowling and G. Mauceri ('86/87),
	A. Nagel, A. Seeger, S. Wainger ('93),
	A. Iosevich, E. Sawyer, A, Seeger ('99)

Representation of S as a graph of ϕ

$S \subset \mathbb{R}^3$ smooth, finite type hypersurface; $x^0 \in S$:

By localization near x^0 and application of Euclidean motion of $\mathbb{R}^3$ we may assume: $x^0 = (0,0,0)$, and

$$S = \{(x_1, x_2, \phi(x_1, x_2)) : (x_1, x_2) \in \Omega\},$$

where $\phi \in C^\infty(\Omega)$ s.t. $\phi(0,0) = 0$, $\nabla\phi(0,0) = 0$. If

$$\phi(x_1, x_2) \sim \sum_{j,k=0}^{\infty} c_{jk} x_1^j x_2^k$$

is the Taylor series of ϕ, define the Taylor support of ϕ at $(0,0)$ by

$$\mathcal{T}(\phi) := \{(j,k) \in \mathbb{N}^2 : c_{jk} \neq 0\}.$$

NOTICE: $\mathcal{T}(\phi) \neq \emptyset$, since ϕ is of finite type at the origin!

1. Newton polyhedron:

$$\mathcal{N}(\phi) := \text{conv} \bigcup_{(j,k)\in\mathcal{T}(\phi)} (j,k) + \mathbb{R}^2_+$$

2. Newton distance : $d = d(\phi)$ is given by the coordinate d of the point (d,d) at which the bisectrix $t_1 = t_2$ intersects the boundary of the Newton polyhedron.
3. Principal face $\pi(\phi)$: The face of minimal dimension containing the point (d,d).
4. Principal part of ϕ :

$$\phi_{\text{pr}}(x_1,x_2) := \sum_{(j,k)\in\pi(\phi)} c_{jk} x_1^j x_2^k$$

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Figure 1

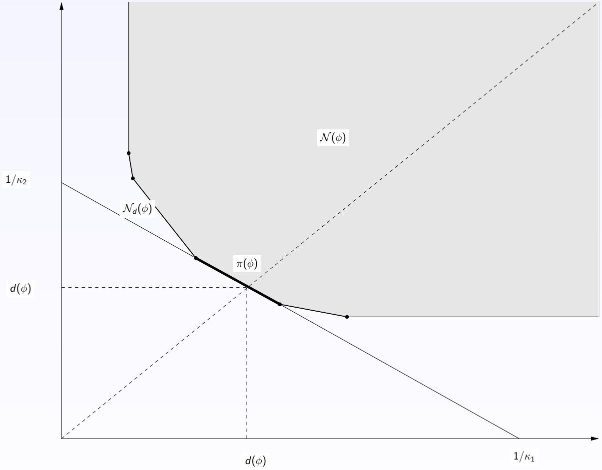

Figure: Newton polyhedron

Adapted coordinates

Height of ϕ :

$$h(\phi) := \sup\{d_x\},$$

where the supremum is taken over all local analytic (resp. smooth) coordinate systems x at the origin, and where d_x is the Newton distance of ϕ when expressed in the coordinates x.

NOTICE: The height is invariant under local smooth changes of coordinates at the origin!

A coordinate system x is said to be adapted to ϕ if $h(\phi) = d_x$.

Example. Let

$$\phi(x_1, x_2) := (x_2 - x_1^m)^n + x_1^\ell.$$

If $\ell > mn$, the coordinates are not adapted. Adapted coordinates are then $y_1 := x_1, y_2 := x_2 - x_1^m$, in which ϕ is given by

$$\phi^a(y) = y_2^n + y_1^\ell.$$

Example 1

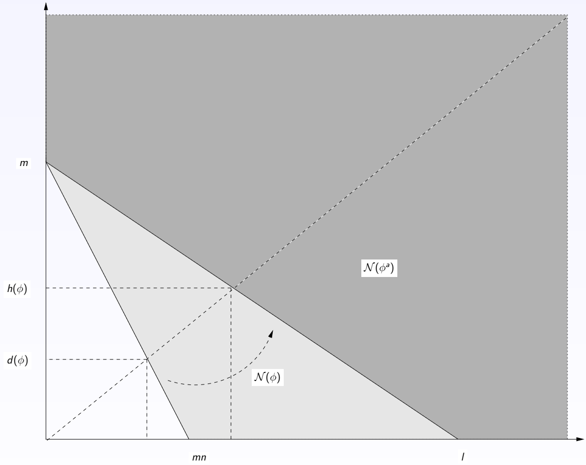

Figure: $\phi(x_1, x_2) := (x_2 - x_1^m)^n + x_1^\ell \quad (\ell > mn)$

Edges and homogeneities

Let $\kappa = (\kappa_1, \kappa_2)$ with, say, $\kappa_2 \geq \kappa_1 > 0$, be a given weight, with corresponding dilations

$$\delta_r(x_1, x_2) := (r^{\kappa_1} x_1, r^{\kappa_2} x_2), \quad r > 0.$$

F on $\mathbb{R}^2$ is κ-homogeneous of degree a, (short: mixed homogeneous) if

$$F(\delta_r x) = r^a F(x) \quad \forall r > 0, x \in \mathbb{R}^2.$$

Choose a so that $L_\kappa := \{(t_1, t_2) \in \mathbb{R}^2 : \kappa_1 t_1 + \kappa_2 t_2 = a\}$ is the supporting line to the Newton polyhedron $\mathcal{N}(\phi)$ of ϕ. The κ-principal part of ϕ

$$\phi_\kappa(x_1, x_2) := \sum_{(j,k) \in L_\kappa} c_{jk} x_1^j x_2^k$$

is κ-homogeneous of degree a.

$$\phi(x_1, x_2) = \phi_\kappa(x_1, x_2) + \text{ terms of higher } \kappa\text{-degree.}$$

NOTICE: If the principal face $\pi(\phi)$ is an edge, then there is a unique weight $\kappa = \kappa_{\mathrm{pr}}$ so that $\pi(\phi)$ lies on the line $\kappa_1 t_1 + \kappa_2 t_2 = 1$.

Adaptedness

Let $P \in \mathbb{R}[x_1, x_2]$ be a κ- homogeneous polynomial with $\nabla P(0,0) = 0$, let

$$m(P) := \mathrm{ord}_{S^1} P$$

be the maximal order of vanishing of P along the unit circle S^1 centered at the origin.

Theorem (Varchenko; Phong, J. Sturm, Stein (analytic ϕ); I.,M.)

There always exist adapted smooth coordinates y, of the form $y_1 = x_1,\ y_2 = x_2 - \psi(x_1)$.

Theorem (Condition for non-adaptedness)

The coordinates x are not adapted to ϕ if and only if the principal face $\pi(\phi)$ of the Newton polyhedron $\mathcal{N}(\phi)$ is a compact edge, and $m(\phi_{\mathrm{pr}}) > d(\phi)$. Moreover, the latter implies that $\frac{\kappa_2}{\kappa_1} \in \mathbb{N}$, where $\kappa := \kappa_{\mathrm{pr}}$.

A. Decay of the Fourier transform of the surface measure

Varchenko's exponent $\nu(\phi) \in \{0, 1\}$: If there exists an adapted local coordinate system y near the origin such that the principal face $\pi(\phi^a)$ of ϕ, when expressed by the function ϕ^a in the new coordinates (i.e. $\phi(x) = \phi^a(y)$), is a vertex, and if $h(\phi) \geq 2$, then we put $\nu(\phi) := 1$; otherwise, we put $\nu(\phi) := 0$.

Theorem

Let $S = \mathrm{graph}(\phi)$ be as before. Then there exists a neighborhood $U \subset S$ of $x^0 = 0$ such that for every $\rho \in C_0^\infty(U)$ the following estimate holds true for every $\xi \in \mathbb{R}^3$:

$$|\widehat{d\mu}(\xi)| \leq C\, \|\rho\|_{C^3(S)} (\log(2 + |\xi|))^{\nu(\phi)} (1 + |\xi|)^{-1/h(\phi)} \tag{3.1}$$

Remarks:

1. In the analytic setting, this is due to V.N. Karpushkin.
2. For ϕ smooth, M. Greenblatt had obtained such estimates for ξ normal to S at 0.

Sharpness

Let N be a unit normal to S at $x^0 = 0$, and put

$$J(\lambda) := \widehat{d\mu}(\lambda N) = \iint e^{\pm i\lambda\phi(x_1,x_2)} a(x_1, x_2)\, dx_1 dx_2, \quad \lambda > 0.$$

Proposition

If in an adapted coordinates system the principal face $\pi(\phi^a)$ is a compact set (i.e. a compact edge or a vertex), then the following limit

$$\lim_{\lambda \to +\infty} \frac{\lambda^{1/h(\phi)}}{\log \lambda^{\nu(\Phi)}} J(\lambda) = C \cdot a(0,0),$$

exists, where C is a non-zero constant depending on ϕ only.

Remarks:

1. This improves on a result by M. Greenblatt, who proved that this limit exists for some sequence of $\lambda_k \to \infty$.
2. If the principal face $\pi(\phi^a)$ is unbounded, then the estimate in the theorem may fail to be sharp, if ϕ is non-analytic, as the following example by A. Iosevich and E. Sawyer shows: If

$$\Phi(x_1, x_2) := x_2^2 + e^{-1/|x_1|^\alpha},$$

then

$$|J(\lambda)| \asymp \frac{1}{\lambda^{1/2} \log \lambda^{1/\alpha}} \quad \text{as} \quad \lambda \to +\infty.$$

Here, $\nu(\phi) = 0$.

B. Sharp estimates for the maximal operator $\mathcal{M}$

- Translations do not commute with dilations.
- $\Longrightarrow$ Euclidean motions are no admissible coordinate changes for the study of the maximal operators $\mathcal{M}$.

Transversality Assumption:

The affine tangent plane $x + T_x S$ to S through x does not pass through the origin in $\mathbb{R}^3$ for every $x \in S$. Equivalently, $x \notin T_x S$ for every $x \in S$, so that $0 \notin S$, and x is transversal to S for every point $x \in S$.

$\Longrightarrow$ If $x^0 \in S$, then there is a **linear** change of coordinates in $\mathbb{R}^3$ so that in the new coordinates $x^0 = 0$, and S is locally given by

$$S = \mathrm{graph}(1 + \phi) \qquad (\phi(0,0) = 0, \nabla\phi(0,0) = 0).$$

Put

$$h(x^0, S) := h(\phi)$$

This notion is invariant under affine linear changes of coordinates in the ambient space $\mathbb{R}^3$!

Maximal estimates

Let $S \subset \mathbb{R}^3$be a hypersurface as before, and $x^0 \in S$. Recall that

$$A_t f(x) := \int_S f(x - ty)\rho(y)\, d\sigma(y), \quad t > 0,$$
$$\mathcal{M}f(x) := \sup_{t>0} |A_t f(x)|.$$

Theorem (Boundedness of $\mathcal{M}$ for $p > 2$)

(i) *Assume that $p > 2$. If the measure $\rho d\sigma$ is supported in a sufficiently small neighborhood of x^0, then $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^3)$ whenever $p > h(x^0, S)$.*

(ii) *If $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^3)$ for some $p > 1$, and if $\rho(x^0) > 0$, then $p \geq h(x^0, S)$. Moreover, if S is analytic at x^0, then $p > h(x^0, S)$.*

Order of contact with hyperplanes

- H affine hyperplane: $\quad d_H(x) := \mathrm{dist}\,(H, x).$

Theorem (Iosevich-Sawyer)

If the maximal operator $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^n)$, where $p > 1$, then

$$\int_S d_H(x)^{-1/p}\,\rho(x)\,d\sigma(x) < \infty \tag{4.2}$$

for every affine hyperplane H in $\mathbb{R}^n$ which does not pass through the origin.

Conjecture (Iosevich-Sawyer)

For $p > 2$ condition (4.2) is necessary and sufficient for the boundedness of $\mathcal{M}$ on L^p.

Theorem

Assume that $S \subset \mathbb{R}^3$ is as before, and ρ is supported in a sufficiently small neighborhood of x^0. If S is analytic, then the conjecture of Iosevich-Sawyer holds true, and if S is only of finite type, then it is true, with the possible exception of the exponent $p = h(x^0, S)$.

Oscillation, order of contact and sublevel estimates

Given $x^0 \in S$, call

1. uniform oscillation index $\beta_u(x^0)$: the supremum over all β s.t.

$$|\widehat{\rho d\sigma}(\xi)| \le C_\beta \, (1 + |\xi|)^{-\beta} \qquad \forall \xi \in \mathbb{R}^n \tag{4.3}$$

for all ρ supported in a sufficiently small neighborhood of x^0.

2. uniform contact index $\gamma_u(x^0)$: the supremum over all γ s.t.

$$\int_S d_H(x)^{-\gamma} \, \rho(x) \, d\sigma(x) < \infty \tag{4.4}$$

for every affine hyperplane H and ρ as before.

3. If we restrict directions to the normal to S in x^0, respectively H to the affine tangent plane in x^0, we introduce accordingly the oscillation index $\beta(x^0)$ and the contact index $\gamma(x^0)$.

Combining our results with results by Phong, Stein and Sturm, we get

Theorem

Let $x^0 \in S \subset \mathbb{R}^3$ be a fixed point. Then

$$\beta_u(x^0, S) = \beta(x^0, S) = \gamma_u(x^0, S) = \gamma(x^0, S) = 1/h(x^0, S).$$

Note: The contact order estimates are essentially equivalent to certain sublevel estimates (Tschebychev!)

The case $p \leq 2$

- If $p \leq 2$, then neither the notion of height nor that of contact index will determine the range of exponents p for which the maximal operator $\mathcal{M}$ is L^p-bounded.
- We have a conjecture for this case, which for certain surfaces relates to fundamental open problems in Fourier analysis, such as the **conjectured reverse square function estimate for the cone multiplier**
- Work in progress!

C. Fourier restriction: Adapted coordinates

We may assume that

$$S = \{(x_1, x_2, \phi(x_1, x_2)) : (x_1, x_2) \in \Omega\}, \quad x^0 = 0.$$

Theorem

Assume that the coordinates (x_1, x_2) are adapted to ϕ, where ϕ is smooth of finite type. If the support of $\rho \geq 0$ is contained in a sufficiently small neighborhood of 0, then

$$\Big(\int_S |\widehat{f}|^2 \rho d\sigma\Big)^{1/2} \leq C_p \|f\|_{L^p(\mathbb{R}^3)}, \qquad f \in \mathcal{S}(\mathbb{R}^3), \tag{5.1}$$

for every $p \geq 1$ such that $p' \geq 2h(\phi) + 2$.

Remarks:

1. Knapp type examples show that our result is sharp.
2. **A. Magyar** had obtained partial results in the analytic case before.

On the proof

1. For $p' > 2h(\phi) + 2$, this follows directly from our Fourier decay estimate (3.1) and

Theorem (Greenleaf '81 - the case $n = 3$)

Assume that $\widehat{d\mu}(\xi) \lesssim |\xi|^{-1/h}$. Then the restriction estimate

$$\Big(\int_S |\widehat{f}|^2 \, d\mu\Big)^{1/2} \leq C_p \|f\|_{L^p(\mathbb{R}^3)}$$

holds for every $p \geq 1$ such that $p' \geq 2h + 2$.

2. The endpoint $p' = 2h(\phi) + 2$ can be obtained by Littlewood-Paley theory.

C. Fourier restriction: Non-adapted coordinates

Then $\pi(\phi)$ is a compact edge, lying on a unique line

$$L := \{(t_1, t_2) \in \mathbb{R}^2 : \kappa_1 t_1 + \kappa_2 t_2 = 1\}.$$

Moreover,

$$m := \kappa_2/\kappa_1 \in \mathbb{N}, \qquad (\text{and } m \geq 2), \tag{5.2}$$

and $m(\phi_{\mathrm{pr}}) > d(\phi)$, so that there is (exactly) one real root $x_2 = b_1 x_1^m$ of ϕ_{pr} of multiplicity bigger than $h(\phi)$, the principal root. Changing coordinates

$$y_1 := x_1, \; y_2 := x_2 - b_1 x_1^m,$$

we arrive at a "better" coordinate system y. By iterating this procedure, we arrive at Varchenko's algorithm for constructing an adapted coordinate system (**in higher dimension, adapted coordinates may not exist!**).

In the end, one can find a change of coordinates

$$y_1 := x_1,\ y_2 := x_2 - \psi(x_1) \tag{5.3}$$

leading to adapted coordinates y for ϕ, where the principal root jet ψ has a Taylor approximation

$$\psi(x_1) = b_1 x_1^m + O(x_1^{m+1}).$$

In the adapted coordinates y, ϕ is given by

$$\phi^a(y) := \phi(y_1, y_2 + \psi(y_1)).$$

Newton polyhedron $\mathcal{N}(\phi^a)$

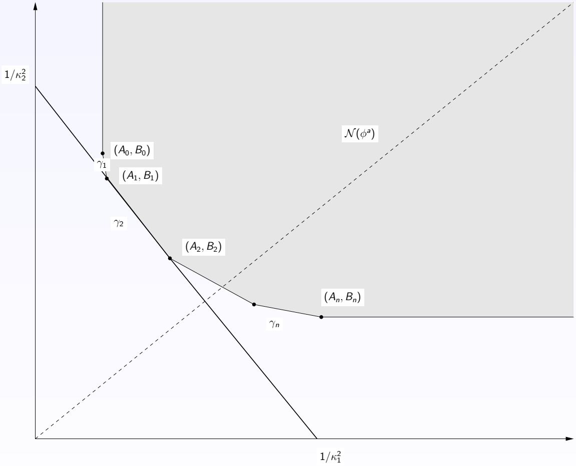

Figure: Edges and weights

m-Height

Consider the line parallel to the bi-sectrix

$$\Delta^{(m)} := \{(t, t+m+1) : t \in \mathbb{R}\}.$$

For any edge $\gamma_\ell \subset L_\ell := \{(t_1, t_2) \in \mathbb{R}^2 : \kappa_1^\ell t_1 + \kappa_2^\ell t_2 = 1\}$ define h_ℓ by

$$\Delta^{(m)} \cap L_\ell = \{(h_\ell - m, h_\ell + 1)\},$$

i.e.

$$h_\ell = \frac{1 + m\kappa_1^\ell - \kappa_2^\ell}{\kappa_1^\ell + \kappa_2^\ell}, \tag{5.4}$$

and define the *m-height of ϕ* by

$$h^{(m)}(\phi) := \max(d, \max_{\ell : a_\ell > m} h_\ell).$$

Remarks:

1. For L in place of L_ℓ, one has $m = \kappa_2/\kappa_1$ and $d = 1/(\kappa_1 + \kappa_2)$, so that one gets d in place of h_ℓ in (5.4)
2. Since $m < a_\ell$, we have $h_\ell < 1/(\kappa_1^\ell + \kappa_2^\ell)$, hence $h^{(m)}(\phi) < h(\phi)$.

m-height $h^{(m)}(\phi)$

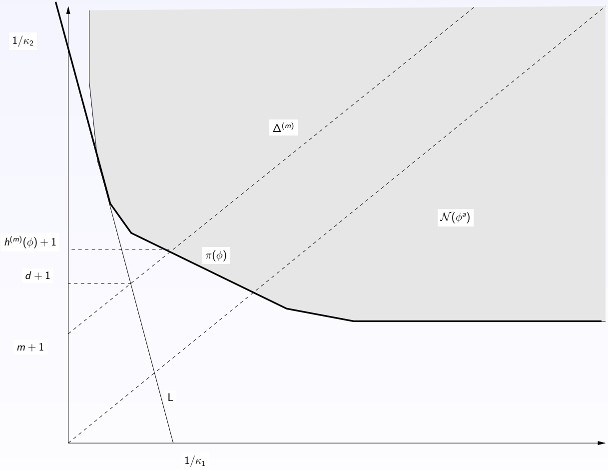

Figure: m-height

Theorem

Assume that there is no linear coordinate system adapted to ϕ, where ϕ is smooth of finite type. Then there exists a neighborhood $U \subset S$ of $x^0 = 0$ such that for every non-negative density $\rho \in C_0^\infty(U)$,

$$\Big(\int_S |\widehat{f}|^2 \rho d\sigma\Big)^{1/2} \le C_p \|f\|_{L^p(\mathbb{R}^3)}, \qquad f \in \mathcal{S}(\mathbb{R}^3), \tag{5.5}$$

for every $p \ge 1$ such that $p' > p'_c := 2h^{(m)}(\phi) + 2$.

Remarks:

1. The condition $p' > 2p'_c + 2$ is weaker than the condition $p' > 2h(\phi) + 2$, which would follow from Greenleaf's result!
2. Again, Knapp type examples show that our result is sharp, except possibly for the endpoint.
3. If ϕ analytic, presumably true also at endpoint $p = p_c$.

Example 2

$$\phi(x_1, x_2) := (x_2 - x_1^m)^n, \qquad n, m \geq 2.$$

The coordinates (x_1, x_2) are not adapted. Adapted coordinates are $y_1 := x_1, y_2 := x_2 - x_1^m$, in which ϕ is given by

$$\phi^a(y_1, y_2) = y_2^n.$$

Here

$$\kappa_1 = \frac{1}{mn}, \quad \kappa_2 = \frac{1}{n},$$
$$d := d(\phi) = \frac{1}{\kappa_1 + \kappa_2} = \frac{nm}{m+1},$$

and

$$p'_c = \begin{cases} 2d+2, & \text{if } n \leq m+1, \\ 2n, & \text{if } n > m+1 \ . \end{cases}$$

Example 2

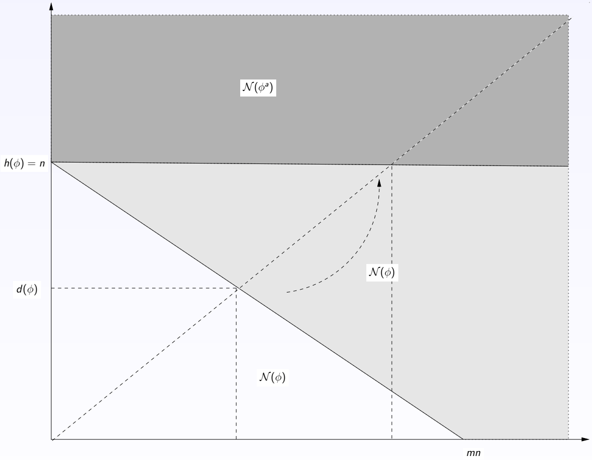

Figure: $\phi(x_1, x_2) := (x_2 - x_1^m)^n \quad (n, m \geq 2)$

BASIC INGREDIENCES OF PROOFS (Maximal Estimate)

- **Puiseux expansion of roots and Newton polyhedra.** Assume ϕ analytic:

$$\phi(x_1,x_2) = U(x_1,x_2)x_1^{\nu_1}x_2^{\nu_2}\prod_r \Big(x_2 - r(x_1)\Big), \quad U(0,0)\neq 0;$$

roots $r(x_1)$ admit a Puiseux series expansion

$$r(x_1) = c_{l_1}^{\alpha_1}x_1^{a_{l_1}} + c_{l_1 l_2}^{\alpha_1\alpha_2}x_1^{a_{l_1 l_2}^{\alpha_1}} + \cdots + c_{l_1\cdots l_p}^{\alpha_1\cdots\alpha_p}x_1^{a_{l_1\cdots l_p}^{\alpha_1\cdots\alpha_{p-1}}} + \cdots ;$$

  - exponents $a_{l_1\cdots l_p}^{\alpha_1\cdots\alpha_{p-1}} > 0$ are all multiples of a fixed rational;
  - $c_{l_1\cdots l_p}^{\alpha_1\cdots\alpha_p} \in \mathbb{C}\setminus\{0\}$.
  - $$a_1 < \cdots < a_\ell < \cdots < a_n$$
    the distinct leading exponents of all the roots r.

Phong and Stein: Group the roots into clusters $[\ell]$ consisting of all roots with leading exponent a_ℓ. Each cluster $[\ell]$ is **associated to an edge** γ_ℓ of $\mathcal{N}(\phi)$.

The ("easy") case when the coordinates are adapted to ϕ (e.g. if ϕ convex)

- **Decomposition**

$$\phi = \phi_{\rm pr} + \text{ error},$$

where $\phi_{\rm pr}$ is κ-homogeneous (if ϕ is convex and finite line type, $\phi_{\rm pr}$ is just the **Schulz polynomial!**). We can then basically reduce to assuming $\phi_{\rm pr} = \phi_\kappa$.

- **Dyadic decomposition and re-scaling** of dyadic pieces using dilations δ_r associated to the weight κ.
- **Control of multiplicity of roots on dyadic pieces:** $\forall x^0$ with $|x^0| \sim 1$ there is a direction e such that

$$\partial_e^m \phi_{\rm pr}(x_1^0, x_2^0) \neq 0 \text{ for some } 2 \leq m \leq h(\phi).$$

- This leads to the right control of oscillatory integrals (**van der Corput!**) or maximal operators , e.g. by reduction to curves:

- **II Decomposition of S into families of curves, e.g. fan decomposition:**

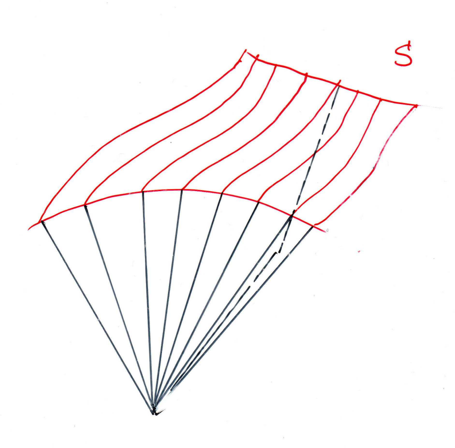

Figure: Fan Decomposition

The case when the coordinates are not adapted to ϕ

III Domain decomposition (no damping technics!)

- $\gamma_1, \gamma_2, \ldots,$ edges of $\mathcal{N}(\phi^a)$ above principal edge, with associated weights κ^ℓ.

Decompose Ω into κ^ℓ-homogeneous domains D_ℓ containing the cluster of non-trivial roots of $\phi^a_{\kappa^\ell}$ associated to γ_ℓ (these roots have multiplicity bounded by $1/(\kappa_1^\ell + \kappa_2^\ell) < h(\phi)$, since they are away from the principal root jet) and the "transition domains" E_ℓ between these domains, which have no homogeneous structure.

- For the domains D_ℓ, one can argue somewhat similarly as in the adapted case, but we also need control on multiplicities of roots of $\partial_2 \phi^a_{\mathrm{pr}}$ and $\partial_2^2 \phi^a_{\mathrm{pr}}$.
- For the transition domains E_ℓ, use bi-dyadic decomposition into rectangles, re-scale, and again reduce, e.g., to maximal averages along curves.

Clusters of roots

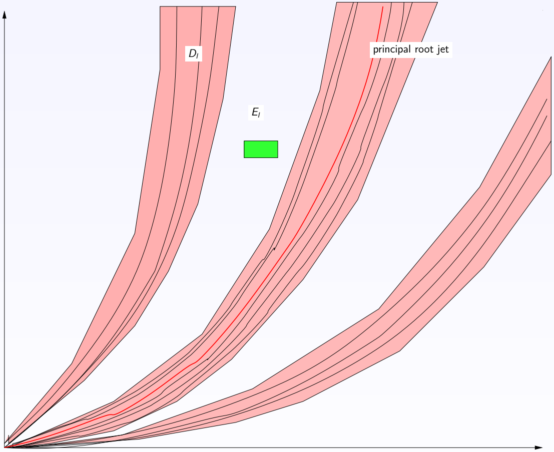

Figure: Clusters of roots

What is left?

- A small, κ^a-homogenous (in the adapted coordinates) neighborhood of the principal root jet ψ which can no longer be dealt with by maximal averages along curves.
- In this domain, in adapted coordinates, the total multiplicity of all roots is controlled by the homogeneous dimension $1/(\kappa_1^a + \kappa_2^a)$ of the principal edge.
- **Main Problem: If** $\partial_2^j \phi_{\mathrm{pr}}^a(y^0) = 0, \ j = 1, \dots, h.$
- To overcome this, e.g. for $\mathcal{M}$ we apply a **further domain decomposition** by means of a stopping time argument into homogeneous domains D'_ℓ and transition domains E'_ℓ, **oriented at the level sets of** $\partial_2 \phi^a$, which again are chopped up into dyadic resp. bi-dyadic pieces.
- After re-scaling, the contributions of these pieces can eventually be estimated by **oscillatory integral technics in 2 variables.**

Oscillatory integrals with small parameters δ, σ

Problem: Oscillatory integrals with small parameters

We need **uniform estimates** of oscillatory integrals of the form

$$J(\xi) = \iint_{\mathbb{R}^2} e^{i(\xi_1 y_1 + \xi_2 \psi(y_1) + \xi_2 y_2 + \xi_3 \phi^a(y))} \eta(y)\, dy,$$

where ϕ^a and ψ depend on **small parameters** and where the interplay between these functions is crucial.

Degenerate Airy type

Most difficult situation: Oscillatory integrals of degenerate Airy type

$$J(\lambda, \sigma, \delta) := \int_{\mathbb{R}^2} e^{i\lambda F(x,\sigma,\delta)} \psi(x, \delta)\, dx,$$

with $F(x_1, x_2, \sigma, \delta) := f_1(x_1, \delta) + \sigma f_2(x_1, x_2, \delta)$.

Example (The following ϕ leads to such oscillatory integrals)

$$\phi(x_1, x_2) := (x_2 - x_1^m)^\ell + x_2 x_1^{n-m},$$

where $n/\ell > m \geq 2$. Here, $\psi(x_1) := x_1^m$,

$$\phi^a(y_1, y_2) = y_1^n + y_2^\ell + y_2 y_1^{n-m},$$

and

$$\phi^a_{\mathrm{pr}}(y_1, y_2) = y_1^n + y_2^\ell.$$

Theorem

Assume that

$$|\partial_1 f_1(0,0)| + |\partial_1^2 f_1(0,0)| + |\partial_1^3 f_1(0,0)| \neq 0$$

and $\partial_1\partial_2 f_2(0,0,0) \neq 0$, *and that there is some* $m \geq 2$ *such that*

$$\partial_2^l f_2(0,0,0) = 0 \text{ for } l = 1, \dots, m-1$$

and $\partial_2^m f_2(0,0,0) \neq 0$.
Then there exist a neighborhood $U \subset \mathbb{R}^2$ *of the origin and constants* $\varepsilon, \varepsilon' > 0$ *such that for any* ψ *which is compactly supported in* U

$$|J(\lambda, \sigma, \delta)| \leq \frac{C\|\psi(\cdot,\delta)\|_{C^3}}{\lambda^{\frac{1}{2}+\varepsilon}|\sigma|^{(l_m + c_m\varepsilon)}},$$

uniformly for $|\sigma| + |\delta| < \varepsilon'$, *where* $l_m := \frac{1}{6}$ *and* $c_m := 1$ *for* $m < 6$, *and* $l_m := \frac{m-3}{2(2m-3)}$ *and* $c_m := 2$ *for* $m \geq 6$.

THANKS FOR YOUR ATTENTION!