Recent developments on the global behavior to critical nonlinear dispersive equations

Carlos E. Kenig

In the last 25 years or so, there has been considerable interest in the study of non-linear partial differential equations, modeling phenomena of wave propagation, coming from physics and engineering. The areas that gave rise to these equations are water waves, optics, lasers, ferromagnetism, general relativity and many others. These equations have also connections to geometric flows, and to Minkowski and Kähler geometries.

Examples of such equations are the generalized KdV equations

$$\begin{cases} \partial_t u + \partial_x^3 u + u^k \partial_x u = 0, & x \in \mathbb{R}, t \in \mathbb{R} \\ u\big|_{t=0} = u_0, \end{cases}$$

the non-linear Schrödinger equations

$$\begin{cases} i\partial_t u + \triangle u + |u|^p u = 0 & x \in \mathbb{R}^N, t \in \mathbb{R} \\ u\big|_{t=0} = u_0 \end{cases} \qquad \text{(NLS)}$$

and the non-linear wave equation

$$\begin{cases} \partial_t^2 u - \triangle u = |u|^p u & x \in \mathbb{R}^N, t \in \mathbb{R} \\ u\big|_{t=0} = u_0 \\ \partial_t u\big|_{t=0} = u_1. \end{cases} \qquad \text{(NLW)}$$

Examples of such equations are the generalized KdV equations

$$\begin{cases} \partial_t u + \partial_x^3 u + u^k \partial_x u = 0, & x \in \mathbb{R}, t \in \mathbb{R} \\ u\big|_{t=0} = u_0, \end{cases}$$

the non-linear Schrödinger equations

$$\begin{cases} i\partial_t u + \triangle u + |u|^p u = 0 & x \in \mathbb{R}^N, t \in \mathbb{R} \\ u\big|_{t=0} = u_0 \end{cases} \qquad \text{(NLS)}$$

and the non-linear wave equation

$$\begin{cases} \partial_t^2 u - \triangle u = |u|^p u & x \in \mathbb{R}^N, t \in \mathbb{R} \\ u\big|_{t=0} = u_0 \\ \partial_t u\big|_{t=0} = u_1. \end{cases} \qquad \text{(NLW)}$$

Examples of such equations are the generalized KdV equations

$$\begin{cases} \partial_t u + \partial_x^3 u + u^k \partial_x u = 0, & x \in \mathbb{R}, t \in \mathbb{R} \\ u\big|_{t=0} = u_0, \end{cases}$$

the non-linear Schrödinger equations

$$\begin{cases} i\partial_t u + \triangle u + |u|^p u = 0 & x \in \mathbb{R}^N, t \in \mathbb{R} \\ u\big|_{t=0} = u_0 \end{cases} \tag{NLS}$$

and the non-linear wave equation

$$\begin{cases} \partial_t^2 u - \triangle u = |u|^p u & x \in \mathbb{R}^N, t \in \mathbb{R} \\ u\big|_{t=0} = u_0 \\ \partial_t u\big|_{t=0} = u_1. \end{cases} \tag{NLW}$$

Inspired by the theory of ordinary differential equations, one defines a notion of well-posedness for these initial value problems, with data u_0 in a given function space B. Since these equations are time-reversible, the intervals of time to be considered are symmetric around the origin.

Well-posedness entails existence, uniqueness of a solution, which describes a continuous curve in the space B, for $t \in I$, the interval of existence, and continuous dependence of this curve on the initial data. If I is finite, we call this local well-posedness; if I is the whole line, we call this global well-posedness.

Inspired by the theory of ordinary differential equations, one defines a notion of well-posedness for these initial value problems, with data u_0 in a given function space B. Since these equations are time-reversible, the intervals of time to be considered are symmetric around the origin.

Well-posedness entails existence, uniqueness of a solution, which describes a continuous curve in the space B, for $t \in I$, the interval of existence, and continuous dependence of this curve on the initial data. If I is finite, we call this local well-posedness; if I is the whole line, we call this global well-posedness.

The first stage of development of the theory concentrated on what I will call the "local theory of the Cauchy problem", which established local well-posedness results on Sobolev spaces B, or global well-posedness for small data in B.

Pioneering works were due to Sigal, Strichartz, Kato, Ginibre–Velo, Pecher, Tsutsumi and many others.

In the late 80's, in collaboration with Ponce and Vega we introduced the systematic use of the machinery of modern harmonic analysis to study the "local theory of the Cauchy problem".

The first stage of development of the theory concentrated on what I will call the "local theory of the Cauchy problem", which established local well-posedness results on Sobolev spaces B, or global well-posedness for small data in B.

Pioneering works were due to Sigal, Strichartz, Kato, Ginibre–Velo, Pecher, Tsutsumi and many others.

In the late 80's, in collaboration with Ponce and Vega we introduced the systematic use of the machinery of modern harmonic analysis to study the "local theory of the Cauchy problem".

The first stage of development of the theory concentrated on what I will call the “local theory of the Cauchy problem”, which established local well-posedness results on Sobolev spaces B, or global well-posedness for small data in B.

Pioneering works were due to Sigal, Strichartz, Kato, Ginibre–Velo, Pecher, Tsutsumi and many others.

In the late 80’s, in collaboration with Ponce and Vega we introduced the systematic use of the machinery of modern harmonic analysis to study the “local theory of the Cauchy problem”.

Further contributions came from work of Bourgain, Klainerman–Machedon, Tataru, Tao and many others.

The resulting body of techniques has proved very powerful in many problems and has attracted the attention of a large community of researchers.

In recent years, there has been a great deal of interest in the study, for non-linear dispersive equations, of the long-time behavior of solutions, for large data. Issues like blow-up, global existence and scattering have come to the forefront, especially in critical problems. These problems are natural extensions of non-linear elliptic problems which were studied earlier.

Further contributions came from work of Bourgain, Klainerman–Machedon, Tataru, Tao and many others.

The resulting body of techniques has proved very powerful in many problems and has attracted the attention of a large community of researchers.

In recent years, there has been a great deal of interest in the study, for non-linear dispersive equations, of the long-time behavior of solutions, for large data. Issues like blow-up, global existence and scattering have come to the forefront, especially in critical problems. These problems are natural extensions of non-linear elliptic problems which were studied earlier.

Further contributions came from work of Bourgain, Klainerman–Machedon, Tataru, Tao and many others.

The resulting body of techniques has proved very powerful in many problems and has attracted the attention of a large community of researchers.

In recent years, there has been a great deal of interest in the study, for non-linear dispersive equations, of the long-time behavior of solutions, for large data. Issues like blow-up, global existence and scattering have come to the forefront, especially in critical problems. These problems are natural extensions of non-linear elliptic problems which were studied earlier.

To explain this connection, recall that in the late 1970's and early 1980's, there was a great deal of interest in the study of semi-linear elliptic equations, to a great degree motivated by geometric applications.

For instance, recall the Yamabe problem:

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Is there a conformal metric $\tilde{g} = cg$, so that the scalar curvature of $(M, \tilde{g})$ is constant?

To explain this connection, recall that in the late 1970's and early 1980's, there was a great deal of interest in the study of semi-linear elliptic equations, to a great degree motivated by geometric applications.

For instance, recall the Yamabe problem:

Let (M, g) *be a compact Riemannian manifold of dimension* $n \geq 3$. *Is there a conformal metric* $\tilde{g} = cg$, *so that the scalar curvature of* $(M, \tilde{g})$ *is constant?*

In this context, the following equation was extensively studied:

$$\triangle u + |u|^{\frac{4}{N-2}} u = 0, \quad \text{for } x \in \mathbb{R}^N, \tag{0.1}$$

where $u \in \dot{H}^1(\mathbb{R}^N) = \{u : \nabla u \in L^2(\mathbb{R}^N)\}$ and $\triangle u = \sum_{j=1}^{N} \frac{\partial^2 u}{\partial x_j^2}$.

Using this information, Trudinger, Aubin and Schoen solved the Yamabe problem in the affirmative.

The equation (0.1) is "critical" because the linear part ($\triangle$) and the non-linear part ($|u|^{4/(n-2))}u$) have the same "strength", since if u is a solution, so is

$$\frac{1}{\lambda^{(N-2)/2}} u\left(\frac{x}{\lambda}\right)$$

and both the linear part and the non-linear part transform in the same way under this change.

The equation (0.1) is "focusing", because the linear part ($\triangle$) and the non-linearity ($|u|^{4/(N-2)}u$) have opposite signs and hence they "fight each other".

The difficulties in the study of (0.1) come from the "lack of compactness" of the Sobolev embedding:

$$\|u\|_{L^{2^*}(\mathbb{R}^N)} \leq C_N \|\nabla u\|_{L^2(\mathbb{R}^N)}, \quad \frac{1}{2^*} = \frac{1}{2} - \frac{1}{N}, \qquad \text{(Sob)}$$

where C_N is the best constant.

The equation (0.1) is "focusing", because the linear part ($\triangle$) and the non-linearity ($|u|^{4/(N-2)}u$) have opposite signs and hence they "fight each other".

The difficulties in the study of (0.1) come from the "lack of compactness" of the Sobolev embedding:

$$\|u\|_{L^{2^*}(\mathbb{R}^N)} \leq C_N \|\nabla u\|_{L^2(\mathbb{R}^N)}, \quad \frac{1}{2^*} = \frac{1}{2} - \frac{1}{N}, \qquad \text{(Sob)}$$

where C_N is the best constant.

The only non-zero radial solution of (0.1) in $\dot{H}^1(\mathbb{R}^N)$ (modulo sign, translation and scaling) and also the only non-negative solution is

$$W(x) = \left(1 + |x|^2/N(N-2)\right)^{-(N-2)/2}$$

(Pohozaev, Gidas–Ni–Nirenberg). W is also the unique minimizer in (Sob) (Talenti).

For the much easier "defocusing problem"

$$\triangle u - |u|^{\frac{4}{N-2}} u = 0, \quad u \in \dot{H}^1(\mathbb{R}^N), \qquad (0.2)$$

it is easy to see that there are no non-zero solutions.

The only non-zero radial solution of (0.1) in $\dot{H}^1(\mathbb{R}^N)$ (modulo sign, translation and scaling) and also the only non-negative solution is

$$W(x) = \left(1 + |x|^2/N(N-2)\right)^{-(N-2)/2}$$

(Pohozaev, Gidas–Ni–Nirenberg). W is also the unique minimizer in (Sob) (Talenti).

For the much easier "defocusing problem"

$$\triangle u - |u|^{\frac{4}{N-2}} u = 0, \quad u \in \dot{H}^1(\mathbb{R}^N), \tag{0.2}$$

it is easy to see that there are no non-zero solutions.

Through the study of (0.1) and (Sob) by Talenti, Trudinger, Aubin, Schoen, Taubes, Schoen-Uhlenbeck, Sachs-Uhlenbeck, Bahri-Coron, Brézis–Coron, etc. many important techniques were developed. In particular, through these works, the study of the “defect of compactness” and the “bubble decomposition” were first understood. These ideas were systematized through P. L. Lions' work on concentration-compactness and other works.

For non-linear dispersive equations there are also “critical problems,” which are related to (0.1), (0.2).

I will now try to describe a program (which I call the concentration-compactness/rigidity theorem method) which Frank Merle and I have developed to study such critical evolution problems in focusing and defocusing cases.

To illustrate the program, we will concentrate on two examples, the "energy critical" non-linear Schrödinger equation and non-linear wave equation:

$$\text{(NLS)} \quad \begin{cases} i\partial_t u + \triangle u \pm |u|^{4/(N-2)}u = 0 & (x,t) \in \mathbb{R}^N \times \mathbb{R} \\ u|_{t=0} = u_0 \in \dot{H}^1(\mathbb{R}^N) \end{cases}$$

$$\text{(NLW)} \quad \begin{cases} \partial_t^2 u - \triangle u = \pm |u|^{4/(N-2)}u & (x,t) \in \mathbb{R}^N \times \mathbb{R} \\ u|_{t=0} = u_0 \in \dot{H}^1(\mathbb{R}^N) \\ \partial_t u|_{t=0} = u_1 \in L^2(\mathbb{R}^N) \end{cases}$$

To illustrate the program, we will concentrate on two examples, the "energy critical" non-linear Schrödinger equation and non-linear wave equation:

$$\text{(NLS)} \quad \begin{cases} i\partial_t u + \triangle u \pm |u|^{4/(N-2)} u = 0 & (x,t) \in \mathbb{R}^N \times \mathbb{R} \\ u|_{t=0} = u_0 \in \dot{H}^1(\mathbb{R}^N) \end{cases}$$

$$\text{(NLW)} \quad \begin{cases} \partial_t^2 u - \triangle u = \pm |u|^{4/(N-2)} u & (x,t) \in \mathbb{R}^N \times \mathbb{R} \\ u|_{t=0} = u_0 \in \dot{H}^1(\mathbb{R}^N) \\ \partial_t u|_{t=0} = u_1 \in L^2(\mathbb{R}^N) \end{cases}$$

In both instances, the "$-$" sign corresponds to the defocusing case, while the "$+$" sign corresponds to the focusing case.

For (NLS) if u is a solution, so is

$$\frac{1}{\lambda^{(N-2)/2}} u\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right),$$

while for (NLW) if u is a solution, so is

$$\frac{1}{\lambda^{(N-2)/2}} u\left(\frac{x}{\lambda}, \frac{t}{\lambda}\right).$$

In both instances, the "$-$" sign corresponds to the defocusing case, while the "$+$" sign corresponds to the focusing case.

For (NLS) if u is a solution, so is

$$\frac{1}{\lambda^{(N-2)/2}} u\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right),$$

while for (NLW) if u is a solution, so is

$$\frac{1}{\lambda^{(N-2)/2}} u\left(\frac{x}{\lambda}, \frac{t}{\lambda}\right).$$

Both scalings leave invariant the norm in the energy spaces $\dot{H}^1$, $\dot{H}^1 \times L^2$, which is why the problems are called "energy critical".

Both problems have "energies" that are constant in time:

$$\text{(NLS)} \qquad E_{\pm}(u_0) = \frac{1}{2}\int |\nabla u_0|^2 \pm \frac{1}{2^*}\int |u_0|^{2^*}$$

$$\text{(NLW)} \qquad E_{\pm}(u_0, u_1) = \frac{1}{2}\int |\nabla u_0|^2 + \frac{1}{2}\int (u_1)^2 \pm \frac{1}{2^*}\int |u_0|^{2^*}$$

where $+$=defocusing case, $-$=focusing case. $\left(\frac{1}{2^*} = \frac{1}{2} - \frac{1}{N}\right)$

Both scalings leave invariant the norm in the energy spaces $\dot{H}^1$, $\dot{H}^1 \times L^2$, which is why the problems are called "energy critical".

Both problems have "energies" that are constant in time:

$$\text{(NLS)} \qquad E_{\pm}(u_0) = \frac{1}{2}\int |\nabla u_0|^2 \pm \frac{1}{2^*}\int |u_0|^{2^*}$$

$$\text{(NLW)} \qquad E_{\pm}(u_0, u_1) = \frac{1}{2}\int |\nabla u_0|^2 + \frac{1}{2}\int (u_1)^2 \pm \frac{1}{2^*}\int |u_0|^{2^*}$$

where $+$=defocusing case, $-$=focusing case. $\left(\frac{1}{2^*} = \frac{1}{2} - \frac{1}{N}\right)$

For both problems the “local theory of the Cauchy problem” has been long understood. (For (NLS) through work of Cazenave–Weissler (1990), for (NLW) through works of Pecher (1984), Ginibre–Velo (1995)).

These works (say for (NLS)) show that for any $u_0 \in \dot{H}^1(\mathbb{R}^N)$,

$$\|u_0\|_{\dot{H}^1} < \delta,$$

$\exists !$ solution of (NLS), defined for all time, depending continuously on u_0 and which scatters,
i.e. $\exists \ u_0^{\pm} \in \dot{H}^1$ such that

$$\lim_{t \to \pm\infty} \|u(t) - w_{\pm}(t)\|_{\dot{H}^1} = 0,$$

where $w_{\pm}$ solves the linear problem

$$\begin{cases} i\partial_t w_{\pm} + \triangle w_{\pm} = 0 \\ w_{\pm}|_{t=0} = u_0^{\pm} \end{cases}.$$

Moreover, given any data u_0 in the energy space, there exist $T_{\pm}(u_0)$ such that there exists a unique solution $u \in C((-T_-(u_0), T_+(u_0)); \dot{H}^1) \cap X$ (where X is a suitable function space) and the interval is maximal. Thus, if say $T_+(u_0)$ is finite and t_n is a sequence converging to T_+, $u(t_n)$ has no convergent subsequence in $\dot{H}^1$. Corresponding results hold for (NLW).

The natural conjecture in defocusing cases (when the linear operator and the non-linearity cooperate) is:

(†) Global regularity and well-posedness conjecture:
The same global result as above holds for large data, i.e. we have global in time well-posedness and scattering for arbitrary data in $\dot{H}^1$ *(*$\dot{H}^1 \times L^2$*). Moreover more regular data preserve this regularity for all time.*

(†) was first established for (NLW) through work of Struwe (1988) in the radial case, Grillakis (1990) in the general case and in this form by Shatah–Struwe (1993,1994), Kapitanski (1993) and Bahouri–Shatah (1998).

The first progress on (†) for (NLS) was due to Bourgain (1999) (radial case $N = 3, 4$). This was followed by work by Tao (2004) (radial case $N \geq 5$), Colliander–Keel–Staffilani–Takaoka–Tao (general case $N = 3$) (2005), Ryckman–Vişan (2006) ($N = 4$), Vişan (2006) ($N \geq 5$).

(†) was first established for (NLW) through work of Struwe (1988) in the radial case, Grillakis (1990) in the general case and in this form by Shatah–Struwe (1993,1994), Kapitanski (1993) and Bahouri–Shatah (1998).

The first progress on (†) for (NLS) was due to Bourgain (1999) (radial case $N = 3, 4$). This was followed by work by Tao (2004) (radial case $N \geq 5$),
Colliander–Keel–Staffilani–Takaoka–Tao (general case $N = 3$) (2005), Ryckman–Vişan (2006) ($N = 4$), Vişan (2006) ($N \geq 5$).

In the focusing case, (†) fails. In fact, for (NLW), H. Levine (1974) showed that if

$$(u_0, u_1) \in H^1 \times L^2, \; E(u_0, u_1) < 0,$$

then $T_{\pm}(u_0, u_1)$ are finite. (This was done through an "obstruction" type of argument).

Recently, Krieger–Schlag–Tataru (2007) constructed explicit radial examples, $N = 3$, with $T_{\pm}(u_0, u_1)$ finite.

In the focusing case, (†) fails. In fact, for (NLW), H. Levine (1974) showed that if

$$(u_0, u_1) \in H^1 \times L^2, \; E(u_0, u_1) < 0,$$

then $T_{\pm}(u_0, u_1)$ are finite. (This was done through an "obstruction" type of argument).

Recently, Krieger–Schlag–Tataru (2007) constructed explicit radial examples, $N = 3$, with $T_{\pm}(u_0, u_1)$ finite.

For (NLS) a classical argument (first discovered by Zakharov and then independently by Glassey (1977)) shows that if

$$\int |x|^2 |u_0(x)|^2 < \infty, \; u_0 \in \dot{H}^1, \; E(u_0) < 0,$$

then $T_\pm(u_0)$ are finite. Moreover,

$$W(x) = \left(1 + |x|^2/N(N-2)\right)^{-(N-2)/2} \in \dot{H}^1$$

and is a static solution of (NLS), (NLW) since

$$\triangle W + |W|^{4/(N-2)} W = 0.$$

Thus scattering need not happen for global solutions.

We now have, for focusing problems:

(††) Ground state conjecture:
There exists a "ground state", whose energy is a threshold for global existence and scattering.

The method that Merle and I have developed gives a "road map" to attack both (†) and for the first time (††). Let us illustrate it with (††) for (NLS), (NLW).

We now have, for focusing problems:

(††) Ground state conjecture:
There exists a "ground state", whose energy is a threshold for global existence and scattering.

The method that Merle and I have developed gives a "road map" to attack both (†) and for the first time (††). Let us illustrate it with (††) for (NLS), (NLW).

Theorem A (Kenig-Merle 2006)

For the focusing energy critical (NLS) $3 \leq N \leq 5$, $u_0 \in \dot{H}^1$, *radial, such that* $E(u_0) < E(W)$, *we have:*

i) *If* $\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$, *the solution exists for all time and scatters.*

ii) *If* $\|u_0\|_{\dot{H}^1} > \|W\|_{\dot{H}^1}$, $T_\pm(u_0) < \infty$

Theorem B (Kenig-Merle 2007)

For the focusing energy critical (NLW), $3 \leq N \leq 5$*,* $(u_0, u_1) \in \dot{H}^1 \times L^2$*,* $E(u_0, u_1) < E(W, 0)$*, we have*

i) *If* $\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$*, the solution exists for all time and scatters.*

ii) *If* $\|u_0\|_{\dot{H}^1} > \|W\|_{\dot{H}^1}$*,* $T_\pm(u_0, u_1) < \infty$

Remark:

There is no radial assumption on Theorem B. Also the case

$$E(u_0, u_1) < E(W, 0), \ \|u_0\|_{\dot{H}^1} = \|W\|_{\dot{H}^1}$$

is impossible by variational estimates (similarly for (NLS)). This proves (††), the ground state conjecture, for (NLW). It was the first full proof of (††) in a significant example.

Killip–Vişan (2008) have combined the ideas in Theorem B with another important new idea, to extend Theorem A to the non-radial case $N \geq 5$.

Remark:

There is no radial assumption on Theorem B. Also the case

$$E(u_0, u_1) < E(W, 0), \ \|u_0\|_{\dot{H}^1} = \|W\|_{\dot{H}^1}$$

is impossible by variational estimates (similarly for (NLS)). This proves (††), the ground state conjecture, for (NLW). It was the first full proof of (††) in a significant example.

Killip–Vișan (2008) have combined the ideas in Theorem B with another important new idea, to extend Theorem A to the non-radial case $N \geq 5$.

“The road map” (applied to the proof of Thm B, i)):

a) Variational arguments (Only needed in focusing cases)

These are “elliptic arguments” which come from the characterization of W as the minimizer of

$$\|u\|_{L^{2^*}} \leq C_N \|\nabla u\|_{L^2}.$$

They yield that if we fix $\delta_0 > 0$ so that

$$E(u_0, u_1) < (1 - \delta_0) E(W, 0)$$

and if

$$\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1},$$

then

$$\|u_0\|_{\dot{H}^1} < (1 - \overline{\delta}) \|W\|_{\dot{H}^1} \qquad \text{(energy trapping)}$$

and

$$\int |\nabla u_0|^2 - |u_0|^{2^*} \geq \overline{\delta} \int |\nabla u_0|^2 \qquad \text{(coercivity).}$$

From this, using preservation of energy and continuity of the flow we can see that for $t \in (-T_-, T_+) = I$,

$$E(u(t), \partial_t u(t)) \approx \|(u(t), \partial_t u(t))\|^2_{\dot{H}^1 \times L^2} \approx \|(u_0, u_1)\|^2_{\dot{H}^1 \times L^2},$$

so that

$$\sup_{t \in I} \|(u(t), \partial_t u(t))\|_{\dot{H}^1 \times L^2} < \infty.$$

This need not guarantee $I = (-\infty, +\infty)$, since, for instance, the Krieger–Schlag–Tataru example has this property.

b) Concentration-compactness procedure

If $E(u_0, u_1) < E(W, 0)$, $\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$, by a)

$$E(u(t), \partial_t u(t)) \approx \|(u_0, u_1)\|^2_{\dot{H}^1 \times L^2}.$$

Thus, if $E(u_0, u_1)$ is small, by the "local Cauchy problem" we have global existence and scattering.
Hence, there is a critical level of energy E_C,

$$0 < \eta_0 \leq E_C \leq E(W, 0)$$

such that if

$$E(u_0, u_1) < E_C, \ \|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1},$$

we have global existence and scattering and E_C is optimal with this property.

Theorem B i) is the statement $E_C = E(W, 0)$. If $E_C < E(W, 0)$, we reach a contradiction by proving:

Proposition (Existence of critical elements)

There exists $(u_{0,C}, u_{1,C})$ *with* $E(u_{0,C}, u_{1,C}) = E_C$, $\|u_{0,C}\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$, *such that, either* I *is finite or, if* I *is infinite, the corresponding solution* u_C *does not scatter. We call* u_C *a "critical element".*

To establish Proposition 1, we need to face the "lack of compactness" and the criticality of the problem. To overcome this we use a "profile decomposition" (first proved for the wave equation by Bahouri-Gérard (1999) and for the $N = 2$ mass critical (NLS) by Merle-Vega (1998)), which is the analog, for wave and dispersive equations, of the elliptic "bubble decomposition".

Proposition (Compactness of critical elements)

$\exists\ \lambda(t) \in \mathbb{R}^+$, $x(t) \in \mathbb{R}^N$, $t \in I$ such that

$$K = \left\{ \left(\frac{1}{\lambda(t)^{\frac{(N-2)}{2}}} u_C \left(\frac{x - x(t)}{\lambda(t)}, t \right), \frac{1}{\lambda(t)^{\frac{N}{2}}} \partial_t u_C \left(\frac{x - x(t)}{\lambda(t)}, t \right) \right) \right. \\ \left. : t \in I \right\}$$

has compact closure in $\dot{H}^1 \times L^2$.

This boils down to the fact that the optimality of E_C forces critical elements to have only 1 "bubble" in their "bubble decomposition".

Proposition (Compactness of critical elements)

$\exists\ \lambda(t) \in \mathbb{R}^+$, $x(t) \in \mathbb{R}^N$, $t \in I$ such that

$$K = \left\{ \left(\frac{1}{\lambda(t)^{\frac{(N-2)}{2}}} u_C\left(\frac{x - x(t)}{\lambda(t)}, t\right), \frac{1}{\lambda(t)^{\frac{N}{2}}} \partial_t u_C\left(\frac{x - x(t)}{\lambda(t)}, t\right) \right) : t \in I \right\}$$

has compact closure in $\dot{H}^1 \times L^2$.

This boils down to the fact that the optimality of E_C forces critical elements to have only 1 “bubble” in their “bubble decomposition”.

Finally, the contradiction comes from:

c) Rigidity Theorem:
If $\overline{K}$ (corresponding to a solution u) is compact,

$$E(u_0, u_1) < E(W, 0),$$

$$\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1},$$

then $(u_0, u_1) = (0, 0)$.

c) clearly gives a contradiction since $E(u_{0,C}, u_{1,C}) = E_C \geq \eta_0 > 0$.

The "road map" has already found an enormous range of applicability to previously intractable problems, in work of many researchers. Many more applications are expected.

Finally, the contradiction comes from:

c) Rigidity Theorem:
If $\overline{K}$ (corresponding to a solution u) is compact,

$$E(u_0, u_1) < E(W, 0),$$

$$\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1},$$

then $(u_0, u_1) = (0, 0)$.

c) clearly gives a contradiction since
$E(u_{0,C}, u_{1,C}) = E_C \geq \eta_0 > 0$.

The "road map" has already found an enormous range of applicability to previously intractable problems, in work of many researchers. Many more applications are expected.

These ideas have also been crucial in recent works of Duyckaerts, Kenig and Merle on the study of "blow-up solutions" to (NLW). Let us consider the case $N = 3$. The example of Krieger-Schlag-Tataru mentioned earlier shows that, for each $\eta_0 > 0$, there exists a radial solution, u, to (NLW), with (say) $T_+ = 1$,

$$\sup_{t\in[0,1)} \|\nabla u(t)\|_{L^2}^2 + \|\partial_t u(t)\|_{L^2}^2 \leq \|\nabla W\|_{L^2}^2 + \eta_0$$

and such that

$$u(t) = \frac{1}{\lambda^{1/2}(t)} W(\frac{x}{\lambda(t)}) + \epsilon(t),$$

where

$$\lim_{t\longrightarrow 1}\int_{|x|<1-t}|\nabla\epsilon(t)|^2+\int_{|x|<1-t}(\partial_t\epsilon(t))^2+\int_{|x|<1-t}\epsilon^6(t)dt=0$$

and $\lambda(t)=(1-t)^{1+r}$, for any $r>1/2$. (The restriction $r>1/2$ is technical, $r>0$ is expected.)

Duyckaerts, Kenig and Merle have recently shown that this behavior is universal: if u is any solution with

$$T_+ = 1,$$

$$\sup_{0<t<1} \|\nabla u(t)\|_{L^2}^2 + \|\partial_t u(t)\|_{L^2}^2 \le \|\nabla W\|_{L^2}^2 + \eta_0$$

where $\eta_0 > 0$ is sufficiently small, then, after a rotation and translation of $\mathbb{R}^3$, there is a small real parameter l and $\lambda(t) > 0, x(t) \in \mathbb{R}^3$, such that

$$u(t) = \pm W_l \left(0, \frac{\cdot - x(t)}{\lambda(t)^{1/2}}\right) \frac{1}{\lambda(t)^{\frac{1}{2}}} + \epsilon(t),$$

where $\epsilon(t)$ is as above and

$$\lim_{t \longrightarrow 1} \frac{\lambda(t)}{1-t} = 0,$$

$$\lim_{t \longrightarrow 1} \frac{x(t)}{1-t} = l\mathbf{e_1}, \ \ \mathbf{e_1} = (1,0,0), |l| \lesssim \sqrt{\eta_0},$$

and

$$W_l(t,x) = W\left(\frac{x_1 - tl}{\sqrt{1-l^2}}, x_2, x_3\right) = \left(1 + \frac{(x_1 - tl)^2}{3\sqrt{1-l^2}} + x_2^2 + x_3^2\right)^{\frac{1}{2}}$$

is the Lorentz transform of W.

Remark:

If u is radial, $l = 0$, $W_l = W$ and $x(t) \equiv 0$.

Moreover, these ideas also lead to the construction of blow-up solutions ($T_+ = 1$) so that

$$\lim_{t \uparrow 1} \|\nabla u(t)\|_{L^2}^2 + \|\partial_t u(t)\|_{L^2}^2 = +\infty.$$

These results also open up many new directions for future research.

Remark:

If u is radial, $l = 0$, $W_l = W$ and $x(t) \equiv 0$.

Moreover, these ideas also lead to the construction of blow-up solutions ($T_+ = 1$) so that

$$\lim_{t \uparrow 1} \|\nabla u(t)\|_{L^2}^2 + \|\partial_t u(t)\|_{L^2}^2 = +\infty.$$

These results also open up many new directions for future research.

Remark:

If u is radial, $l = 0$, $W_l = W$ and $x(t) \equiv 0$.

Moreover, these ideas also lead to the construction of blow-up solutions ($T_+ = 1$) so that

$$\lim_{t \uparrow 1} \|\nabla u(t)\|_{L^2}^2 + \|\partial_t u(t)\|_{L^2}^2 = +\infty.$$

These results also open up many new directions for future research.

Thank you for your attention!