Extremizers and Near-extremizers for the Radon Transform

—

A Tale of Three Operators

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Radon Transform

The Radon transform takes a function

$$f : \mathbb{R}^d \to \mathbb{C}$$

and transforms it to a function

$$\mathcal{R}f : G_d \to \mathbb{C}$$

where $G_d =$ Grassmann manifold of all **affine** hyperplanes $\pi \subset \mathbb{R}^d$ by

$$\mathcal{R}(\pi) = \int_\pi f.$$

A Radon Transform Inequality

- Identify Grassmann manifold G_d with $\{(r, \theta) \in \mathbb{R} \times S^{d-1}\}$; $(r, \theta) \leftrightarrow$ hyperplane with normal vector θ, at signed distance r from 0. Essentially a two to one map.
- Equip G_d with measure $d\mu = dr\, d\theta$.
- $\mathcal{R}$ maps $L^1(\mathbb{R}^d)$ to $L^1(G_d)$. (Trivial; Fubini)
- $\mathcal{R}$ maps $L^p(\mathbb{R}^d)$ to $L^q(G_d)$ for

  $$\mathbf{p} = \frac{\mathbf{d+1}}{\mathbf{d}} \text{ and } \mathbf{q} = \mathbf{d+1}.$$

  ([Oberlin-Stein 1981], [Calderón 1983], ...)
- A one-parameter family of inequalities follows from these two. No others are true.
- Therefore: most interesting inequality is $L^{(d+1)/d} \to L^{d+1}$.

Inverse Problems

Let $\mathbb{A}$ be the optimal constant in the inequality

$$\|\mathcal{R}f\|_{d+1} \leq \mathbb{A}\|f\|_{(d+1)/d}.$$

What is the nature of those functions which **extremize**, or nearly extremize, or extremize it to within a small factor?

In this talk I will

- First review results on various versions of this question, obtained over several years.
- Then focus on a result obtained in recent weeks.

Symmetry

- The $L^{(d+1)/d} \to L^{d+1}$ inequality for $\mathcal{R}$ has an extraordinarily

  large group of symmetries.

- Indeed,

$$\frac{\|\mathcal{R}(f \circ \phi)\|_{d+1}}{\|f \circ \phi\|_{(d+1)/d}} = \frac{\|\mathcal{R}(f)\|_{d+1}}{\|f\|_{(d+1)/d}}$$

  for any invertible **affine transformation** ϕ of $\mathbb{R}^d$.

- This high degree of symmetry gives the inequality a special interest.

The endpoint inequality is not particularly difficult to prove. One proof (Calderón, Oberlin-Stein circa 1981) uses the L^2 identity $\mathcal{R}^* \circ \mathcal{R} = (-\Delta)^{-(d-1)/4}$ together with Sobolev embedding, mixed norm spaces, and interpolation.

Quasiextremals

A combinatorial proof ([C 2006, 2011]) leads naturally to this conclusion:

Theorem. [C 2006/2011] If $\|\mathcal{R}\mathbf{1}_E\|_{d+1} \geq \delta\|\mathbf{1}_E\|_{(d+1)/d}$ then

there exists a **convex** set $\mathcal{C}$

which is not large:

$$|\mathcal{C}| = |\mathbf{E}|,$$

but which contains a significant fraction of E:

$$|\mathcal{C} \cap \mathbf{E}| \geq \mathbf{c}\delta^{\mathbf{C}}|\mathbf{E}|.$$

Here c, C are constants which depend only on the dimension d.

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- A natural way to prove this is by establishing **(Pre)compactness of extremizing sequences**:
  If

$$\|f_n\|_{(d+1)/d} = 1 \quad \text{and} \quad \|\mathcal{R}f_n\|_{d+1} \to \mathbb{A}.$$

  then

  Some subsequence converges in $L^{(d+1)/d}$ norm.

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Theorem. Suppose $\|f_n\|_{(d+1)/d} = 1$ and $\|\mathcal{R}f_n\|_{d+1} \to \mathbb{A}$. Then

there exist invertible affine transformations $\phi_n \in G$

such that some subsequence of $\{c_n\, f_n \circ \phi_n\}$ converges, in $L^{(d+1)/d}$ norm.

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Corollary. (**Near-extremizers.**) If

$$\|\mathcal{R} f\|_{d+1} \geq (\mathbb{A} - \varepsilon)\|f\|_{(d+1)/d},$$

then f is $\delta(\varepsilon)$–close to some extremizer.

Properties of Extremizers

- Extremizers are critical points of the functional

$$\mathbf{\Phi}_{\mathcal{R}}(\mathbf{f}) = \frac{\|\mathcal{R}f\|_{d+1}}{\|f\|_{(d+1)/d}}.$$

- Critical points are characterized by a generalized Euler-Lagrange equation:

$$\mathbf{f} = \lambda\Big(\mathcal{R}^*(\mathcal{R}\mathbf{f})^{\mathbf{d}}\Big)^{\mathbf{d}}.$$

- $\lambda \in \mathbb{R}$ is a Lagrange multiplier (a ratio of powers of norms of $\mathcal{R}f$ and f).
- $\mathcal{R}^*$ is the transpose of $\mathcal{R}$ — a similar operator.
- **Theorem.** [C-Xue 2010] All critical points of $\Phi_{\mathcal{R}}$, and in particular all extremizers, are C^∞, and $f(x) \to 0$ as $|x| \to \infty$.

Asymptotic Behavior of Critical Points at Infinity

Theorem. [C, May 2011]
Any critical point f of the functional $\Phi_{\mathcal{R}}$ admits an asymptotic expansion of the form

$$\mathbf{f}(\mathbf{x}) = \sum_{\mathbf{k}=0}^{\infty} \mathbf{g}_{\mathbf{k}}\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) |\mathbf{x}|^{-\mathbf{d}-\mathbf{k}} \quad \text{as } |\mathbf{x}| \to \infty$$

with each $g_k \in C^{\infty}(S^{d-1})$.

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- There do exist critical points which are not extremizers.
- There exist critical points which tend to zero faster, as $|x| \to \infty$, than extremizers do.

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- Therefore one cannot prove the regularity theorem by a direct "bootstrapping" argument.
- It is not true that arbitrary solutions of the Euler-Lagrange equation are C^∞; **there are nonsmooth solutions in weak $L^{(d+1)/d}$**.
- If time permits, I will briefly discuss the proof of this regularity theorem at the end of today's talk.

Identification of Extremizers

I turn now to the main topic of today's lecture, the identification of extremizers.

Very few nontrivial inequalities have known extremizers

- Young's convolution inequality (Beckner [1975]; Lieb [1976]):

$$\iint_{\mathbb{R}^{d+d}} \mathbf{f(x)g(y)h(x-y)\,dx\,dy} \le \mathbf{A_{p,q,r}}\|\mathbf{f}\|_{\mathbf{p}}\|\mathbf{g}\|_{\mathbf{q}}\|\mathbf{h}\|_{\mathbf{r}}$$

if $p^{-1}+q^{-1}+r^{-1}=2.$ Extremizers are certain triples (f,g,h) of (not necessarily radial) **Gaussians** $e^{-Q(x)}$.

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- Hardy-Littlewood-Sobolev inequality (Lieb [1983]):

$$\iint_{\mathbb{R}^{d+d}} \mathbf{f(x)g(y)|x-y|^{-d+\gamma}\,dx\,dy \le A_{p,q,\gamma}\|f\|_p\|g\|_q}$$

if $q=p$ and $2p^{-1}=1+\gamma d^{-1}$. Extremizers

$$\mathbf{F(x)=(1+|x|^2)^{-d/p}}$$

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  if $p^{-1} + q^{-1} + r^{-1} = 2$. Extremizers are certain triples (f, g, h) of (not necessarily radial) **Gaussians** $e^{-Q(x)}$.

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  if $q = p$ and $2p^{-1} = 1 + \gamma d^{-1}$. Extremizers

  $\mathbf{F(x) = (1 + |x|^2)^{-d/p}}$ and $F \circ \phi$ for all $\phi(x) = a + rx$, $0 \neq r \in \mathbb{R}$.

More Inequalities With Known Extremizers

- **Hausdorff-Young** inequality (Babenko, Beckner). Extremizers are Gaussians.
- **Strichartz** inequalities (Foschi): If

  $$u(t,x) = \int_{\mathbb{R}^d} f(\xi) e^{ix\cdot\xi} e^{it|\xi|^2}\, d\xi$$

  then

  $$\|u\|_q \leq A_d \|f\|_2$$

  for $q = 2\frac{d+1}{d-1}$ and $d = 2$ or $d = 3$. Extremizers are (complex, radial) Gaussians.
- **Hilbert transform** (Pichorides).
- Extremizers and optimal constants have been identified for a few other inequalities, by very interesting **exploitation of symmetries**, by Beckner, in an ongoing series of papers.

Today's Main Result

Theorem.

- The function
$$\mathbf{F(x) = (1 + |x|^2)^{-d/2}}$$
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- Each extremizer is of the form
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- That $(1 + |x|^2)^{-d/2}$ is an extremizer, had been conjectured by Baernstein and Loss [1997].
- They proved this for dimension $d = 3$ (and treated the 2-plane transform for all d).

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- **Additional symmetry.**
  The inequality has additional symmetries, beyond the affine group.

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with coordinates $(u, s) \in \mathbb{R}^{d-1} \times \mathbb{R}^1$.

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  then

$$f(x) = c(1 + r|x|^2)^{-d/2} \text{ for some constants } c, r.$$

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- The general scheme is along the lines introduced by **Lieb** in his work on the Hardy-Littlewood-Sobolev inequality.

A Convolution Operator

Before we go on, I need to tell you about a different operator.

- $T : L^{(d+1)/d}(\mathbb{R}^d) \to L^{d+1}(\mathbb{R}^d)$:

$$Tf(x) = f * \sigma(x) = \int f(x-y)\, d\sigma(y)$$

  where σ is the "affine surface measure" $\mathbf{dy'}$ on the parabola

$$\mathbf{y_d} = \tfrac{1}{2}|\mathbf{y'}|^2 \quad \text{with } \mathbf{y} = (\mathbf{y'}, \mathbf{y_d}) \in \mathbb{R}^{\mathbf{d-1}} \times \mathbb{R}^{\mathbf{1}}.$$

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- T is a canonical representative of the large class of operators defined by convolution with generalized surface measures on submanifolds. All of the theorems about quasiextremals, existence of extremizers, precompactness of extremizing sequences, and regularity of critical points of $\Phi_{\mathcal{R}}$, were originally proved for T.

A Third Operator

- $\mathcal{R}^\sharp : L^{(d+1)/d}(\mathbb{R}^d) \to L^{d+1}(\mathbb{R}^d)$:

$$\mathcal{R}^\sharp f(x) = \int_{\mathbb{R}^{d-1}} f(y', x_d + x' \cdot y')\, dy'.$$

- $\mathcal{R}^\sharp$ is essentially the Radon transform $\mathcal{R}$, but with different normalizations in two different respects.

A rose by any other name ...

- Fact: For all functions f,

$$\|\mathcal{R}f\|_{d+1} = \|Tf\|_{d+1} = \|\mathcal{R}^\sharp f\|_{d+1}.$$

These identities rely on **(nonaffine)** changes of coordinates, Jacobian factors, and the particular exponents $p = \frac{d+1}{d}$ and $q = d + 1$.

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- The equivalence of T with $\mathcal{R}^\sharp$ is based on the quadratic skew shifts

$$(x', x_d) \mapsto (x', x_d \pm \tfrac{1}{2}|x'|^2)$$

  where $(x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}^1$.

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  where $(x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}^1$.

- $\mathcal{R}^\sharp$ is less famous, but is the bridge between T and $\mathcal{R}$. The proof of today's theorem relies on a symmetry which was discovered for $\mathcal{R}^\sharp$, rather than for $\mathcal{R}$.

Step 2: The Non-Affine Symmetry

- $\mathcal{R}^\sharp$ is a scaling limit (under a parabolic rescaling) of $\mathcal{R}$. Yet a casual glance suggests that $\mathcal{R}^\sharp$ enjoys a smaller symmetry group than $\mathcal{R}$. A search for the missing symmetries led to:
- Set

$$Jf(x'', s, t) = |s|^{-d} f(s^{-1}x'', s^{-1}, s^{-1}t)$$
$$Lf(x'', s, t) = f(x'', t, s)$$

-

$$L \circ \mathcal{R}^\sharp = \mathcal{R}^\sharp \circ J.$$

  Thus J arises naturally, being dual to the affine symmetry L.
- The equivalence between $\mathcal{R}^\sharp$ and $\mathcal{R}$ involves a non-affine change of variables.

- All theorems in this talk have versions for all three operators, in particular, for the convolution operator T associated to the paraboloid.
- Among the extremizers for T are all functions

$$c\left(1 + a|x'|^2 + b\left| x_d - \tfrac{1}{2}|x'|^2 \right|\right)^{-d}$$

  with coordinates $(x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}^1$.

We next discuss Step 1: Every extremizer has “ellipsoidal” symmetry.

An Identity (Drury [1983,1984]; C [2011])

$$\|\mathcal{R}f\|_{d+1}^{d+1} = \int_{(\mathbb{R}^{d-1})^{d+1}} \text{Inner integral } (x'_0, \cdots, x'_d) \prod_{j=0}^{d} dx'_j$$

where $x_j = (x'_j, t_j) \in \mathbb{R}^{d-1} \times \mathbb{R}^1$, and the inner integral is

$$\int_{\mathbb{R}^d} \boldsymbol{\Delta}_{\mathbf{d-1}}(\mathbf{x'_1}, \cdots, \mathbf{x'_d})^{-1} \, \mathbf{f}(\mathbf{x'_0}, \mathbf{v}(\mathbf{x'}) \cdot \mathbf{t}) \prod_{\mathbf{j=1}}^{\mathbf{d}} \mathbf{f}(\mathbf{x'_j}, \mathbf{t_j}) \, \mathbf{dt_j}$$

with $t = (t_1, \cdots, t_d)$,
Δ_{d-1} = volume of $d-1$–simplex in $\mathbb{R}^{d-1}$ with indicated vertices,
$v(x') = v(x'_1, \cdots, x'_d)$ = unique vector such that

$(x'_0, v(x') \cdot t)$ **is coplanar with** (x'_j, t_j) for all $j \in \{1, \cdots, d\}$.

Radial Nonincreasing Rearrangement

If $f : \mathbb{R}^d \to [0, \infty)$ is finite a.e. then there exists essentially unique $f^* : \mathbb{R}^d \to [0, \infty)$ satisfying:

- **Radial**
- **Nonincreasing**: $r > r' \Rightarrow f^*(r) \le f^*(r')$
- **Equimeasurable** with f:

$$\left|\{x : f^*(x) > \lambda\}\right| = \left|\{x : f(x) > \lambda\}\right|$$

  for all λ.

Rearrangement Inequalities

- Riesz [1930]; Sobolev [1936]

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} f(x)g(y)h(x+y)\,dx\,dy \le \iint_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{f}^*(x)\mathbf{g}^*(y)\mathbf{h}^*(x+y)\,dx\,dy.$$

- Brascamp-Lieb-Luttinger [1974]

$$\int_{\mathbb{R}^m} \prod_j \mathbf{f_j}(L_j(x))\,dx \le \int_{\mathbb{R}^m} \prod_j \mathbf{f_j^*}(L_j(x))\,dx$$

for any linear mappings $L_j : \mathbb{R}^m \to \mathbb{R}^1$.

Rearrangement and the Radon Transform [C 1984]

- The inner integrals in the Drury/C identity are of the Brascamp-Lieb-Luttinger form.
- **Consequence:** Let $f^{\star\star}(x', x_d)$ be nondecreasing rearrangement of $f(x', x_d)$ **with respect to the x_d variable for each x'**. Then

$$\|\mathcal{R}\mathbf{f}^{\star\star}\|_{\mathbf{d+1}} \geq \|\mathcal{R}\mathbf{f}\|_{\mathbf{d+1}}.$$

- Doing this repeatedly with respect to a dense set of directions in $\mathbb{R}^d$ and extracting limit of a subsequence, one obtains for the **radial** f^*

$$\frac{\|\mathcal{R}f^*\|_{d+1}}{\|f^*\|_{(d+1)/d}} \geq \frac{\|\mathcal{R}f\|_{d+1}}{\|f\|_{(d+1)/d}}.$$

(Justification: Brascamp-Lieb-Luttinger [1974]; Carlen-Loss [1990])

Consequence

Corollary. There exist radial extremizers for our Radon transform inequality.

• Use previously known existence of extremizers for an equivalent problem

OR

• Extremize within class of all radial functions.

Burchard's Inverse Theorem [1996]

If

$$\int_{\mathbb{R}^m} \prod_{j=0}^{m} f_j(L_j(x))\,dx \boxed{=} \int_{\mathbb{R}^m} \prod_{j=0}^{m} f_j^*(L_j(x))\,dx$$

then there exist c_j such that

$$\mathbf{f_j(x) \equiv f_j^*(x - c_j)}$$

and moreover $\{c_j\}$ **are compatible** in the sense that there exists $v \in \mathbb{R}^m$ such that

$$c_j = L_j(v) \text{ for all } j \in \{0, \cdots, m\} \quad —$$

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provided that all level sets of all f_j are null sets: For all λ,

$$\big|\{x : f_j(x) = \lambda\}\big| = 0.$$

Scholium

- Burchard's theorem, as stated, does not apply directly, because it requires that level sets be null sets, which is not obvious here.
- I believe that it is possible to refine the analysis of [C-Xue 2010] to show that all critical points of $\Phi_{\mathcal{R}}$ are real analytic. This implies the null condition.
- Burchard proved a more fundamental inverse theorem, characterizing cases of equality in Young's inequality, when all functions are characteristic functions of sets. When the measures of these sets satisfy certain inequalities, the same characterization of equality holds.
- A tedious direct argument supplements this inverse theorem, allowing one to draw the desired conclusion in our application to the Radon transform. I will spare you all these details.

Upshot for Radon Transform

- **Proposition.** If f is an extremizer for the Radon transform inequality then for each unit vector $v \in \mathbb{R}^d$, there exists a **skew reflection** R_v in the direction v such that

$$f \equiv f \circ R_v.$$

- By a skew reflection R_v I mean an **affine involution** such that $R_v(x) - x$ **is parallel to** v for every x.

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- By a skew reflection R_v I mean an **affine involution** such that $R_v(x) - x$ **is parallel to** v for every x.
- The discussion so far provides a skew reflection R_v for every direction v, but no control over their relationship to one another.
- Burchard faced this same point in extending the inverse theorem to higher dimensions.

By their fruits ye shall know them

How to recognize functions with ellipsoidal symmetry?

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- Consider $G =$ **group of all affine symmetries of** f. This includes the group generated by all the skew reflections of the preceding slide.
- G must be **compact**.

By their fruits ye shall know them

How to recognize functions with ellipsoidal symmetry?

- Consider $G =$ **group of all affine symmetries of** f. This includes the group generated by all the skew reflections of the preceding slide.
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Group Theory!

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  Group Theory!

  Fact. **Any compact subgroup of the affine group, is conjugate (by an element of the affine group) to a subgroup of the orthogonal group** $O(d)$**.**
- Therefore after an affine change of variables, any extremizer f is invariant with respect to (orthogonal) reflection about every codimension one subspace of $\mathbb{R}^d$. Thus f **has ellipsoidal symmetry.**

Step 3

Proposition. If f is radial and $|s|^{-d} f(s^{-1}u, s^{-1})$ is "ellipsoidal" then $f(x) = c(1 + r|x|^2)^{-d/2}$ for some constants c, r.

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  for some unknown function h and numbers a, λ. Writing $f(x) = g(|x|^2)$, get

  $$g(\frac{t^2+1}{s^2}) = s^2 h((s-a)^2 + \lambda t^2).$$

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- Apply V to both sides.

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- Therefore
$$\frac{h(\psi)}{h'(\psi)} \equiv -\psi.$$
- Solve ODE
$$\frac{d}{d\psi} \log h(\psi) = -\frac{1}{\psi}$$
to finish.

Solving the Euler-Lagrange equation: **Broken Symmetry**

- Define $S(f) = \big(T^*(Tf)^d\big)^d$.
- The equation $f = \lambda S(f)$ has critical scaling, viewed as an equation in the class L^{p_0}.
- But no individual function in L^{p_0} has critical scaling in any sense; the choice of a particular function f breaks all symmetry.
- The Euler-Lagrange equation has critical scaling only because of its particular degree of nonlinearity. The linearization of S about any L^{p_0} function is a better operator. Thus

Nonlinearity works to our advantage.

Lightning Synopsis

- Main point is to prove improved decay: Solutions satisfy $\langle x \rangle^{\delta} f \in L^{(d+1)/d}$ for some uncontrollable $\delta = \delta(f) > 0$.

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- Carrying this out requires inventing appropriate scale of Banach spaces, and proving various inequalities for S within this scale.

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- Interpolate between unweighted inequality for $L^{(d+1)/d}$, and weighted inequality for L^∞.
- [C-Xue] had carried this out for the convolution operator T, without having observed the equivalent reformulation in terms of an operator with radial symmetry. The weights used there were powers of $(1 + |x'| + |x_d \pm \frac{1}{2}|x'|)$; the proof of the L^∞ inequality took 6 pages of small print.

Great Leap Strategy

- Let the multilinear operator $\vec{S}$ be the "polarization" of S: $S(f) = \vec{S}(f, f, \cdots, f)$ where

$$\vec{S}\{f_{i,j} : 1 \leq i, j \leq d\} = \prod_i T^*\Big(\prod_j T(f_{i,j})\Big).$$

- Goal: Prove if $f \in L^{p_0}$ satisfies $f = \lambda S(f)$, then $\langle x \rangle^{\delta} f \in L^{p_0}$. $(\delta = \delta(f) > 0.)$
- GL strategy requires inventing **a scale of Banach spaces X_t**, for $t \in [0, 1]$, with certain properties.
- The proof will give no control on δ; it depends on the "profile" of f itself, not merely on λ and $\|f\|_{p_0}$.

Properties required of the spaces X_t

The Banach spaces X_t must satisfy:

- $X_0 = L^{p_0}$.
- For $t > 0$, $X_t \subset \langle x \rangle^{-\delta} X_0$ for some $\delta = \delta(t) > 0$.
- $s > t \Rightarrow X_s \subset X_t$.
- If $f \in X_s$ then $t \mapsto \|f\|_{X_t}$ **is continuous** on $[0, s]$.
- $\vec{S} : X_t \otimes X_t \otimes \cdots \otimes X_t \to X_t$ for all t.
- **Wealth sharing property.** Let $t > 0$. If

$$f_{i,j} \in X_0 \ \forall i, j$$

and at least one function $f_{i,j}$ belongs to X_t,

then

$$\vec{S}\{f_{i,j}\} \text{ belongs to } X_\tau \quad \text{for some } \tau(t) > 0.$$

Preparing to Leap

- For small $\varepsilon > 0$ split
$$f = g_\varepsilon + b_\varepsilon$$
where the **good part** $g_\varepsilon \in X_1$, while the **bad part is small**:
$$\|b_\varepsilon\|_{p_0} < \varepsilon.$$
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- Rewrite equation as $b_\varepsilon = \mathcal{L}(g_\varepsilon, b_\varepsilon) + \lambda S(b_\varepsilon)$, where the "linear" part $\mathcal{L}(g_\varepsilon, b_\varepsilon)$ has degree $d - 1$ as a function of b_ε.

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- By wealth sharing, $\mathcal{L}(g_\varepsilon, b_\varepsilon) \in X_s$ for some $s > 0$.
- By continuity of norms,
$$\|\mathcal{L}(g_\varepsilon, b_\varepsilon)\|_{X_{\tau(\varepsilon)}} < 2\varepsilon$$
provided $\tau(\varepsilon) > 0$ is chosen to be sufficiently small.

The bad term is also good

- The bad part b_ε has small norm.
- The most nonlinear term, $\lambda S(b_\varepsilon)$, is homogeneous of degree $d^2 > 1$.
- Therefore this worst term is $\lll b_\varepsilon$ itself.

The Leap

- For small $\varepsilon > 0$, solve fixed-point equation for unknown h_ε:

$$\mathbf{h}_\varepsilon = \mathcal{L}(g_\varepsilon, b_\varepsilon) + \lambda S(\mathbf{h}_\varepsilon)$$

- Do this by contraction mapping principle, **simultaneously** in X_0 and in $X_{\tau(\varepsilon)}$. **This works for all sufficiently small ε.**
- Contraction mapping produces a unique solution. Since $X_{\tau(\varepsilon)} \subset X_0$, the two solutions obtained in the two spaces must agree.
- b_ε is the unique small solution in X_0.
- Therefore for all sufficiently small ε, $b_\varepsilon \in X_{\tau(\varepsilon)}$. Then $f = b_\varepsilon + g_\varepsilon$ also $\in X_{\tau(\varepsilon)}$.

- The spaces X_t used in the proof are weighted spaces $L^{p_t}(w_t)$ where the weights w_t are appropriate positive powers of $\langle x \rangle$.
- The endpoints are $L^{(d+1)/d} \to L^{d+1}$ with weight $|x|^0$, and L^∞ with weight $\langle x \rangle^d$.
- The Radon transform maps these boundedly to L^{q_t} with corresponding power weights.
- The L^∞ endpoint estimate is one line.
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