BOUNDED L^2 CURVATURE CONJECTURE

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May 17, 2011

ON STEIN'S CONTRIBUTION TO PDE

- Clarified, extended and popularized Calderon-Zygmund theory of singular integrals
- Father of Stein-Thomas-Strichartz inequalities
- Clarified, extended and popularized Littlewwod-Paley theory

LITTLEWOOD-PALEY THEORY (APPROPRIATELY EXTENDED) IS NOT JUST A GOOD IDEA IN HARMONIC ANALYSIS BUT RATHER A TRULY REVOLUTIONARY METHODOLOGY WHICH HAS COMPLETELY TRANSFORMED **NONLINEAR PDE**

Based on a powerful **divide and conquer** strategy which enables the analyst to focus on the main difficulties of the problem at hand while ignoring a multitude of other less essential ones (or equally essential but somehow not interacting with the ones we choose to focus on).

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- III. YANG-MILLS EQUATIONS IN FLAT SPACE
- IV . BILINEAR ESTIMATES IN FLAT SPACE
- V. STRATEGY OF PROOF OF THE CONJECTURE.

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Einstein equation in Vacuum

Solutions: Riccci flat space-times $(\mathcal{M}, \mathbf{g})$.

$$\mathbf{Ric}(\mathbf{g}) = 0.$$

Initial data sets: (Σ_0, g, k) +constraints.

Classical WP Theorem: $g \in H^s_{loc}(\Sigma_0)$, $k \in H^{s-1}_{loc}(\Sigma_0)$, $\boxed{s > 5/2}$
$\Rightarrow$ unique space-time $(\mathcal{M}, \mathbf{g})$ and $(\Sigma_0, g, k) \hookrightarrow (\mathcal{M}, \mathbf{g})$ such that (g, k) are the first and second fundamental forms of Σ_0 in $\mathcal{M}$.

Wave coordinates: $\square_{\mathbf{g}} x^\alpha = 0 \Rightarrow$

$$\mathbf{g}^{\alpha\beta}\, \partial_\alpha \partial_\beta\, \mathbf{g}_{\mu\nu} = F_{\mu\nu}(\mathbf{g}, \partial \mathbf{g})$$

Model equation

$$\mathbf{g}^{\alpha\beta}(\phi)\, \partial_\alpha \partial_\beta\, \phi = F(\phi, \partial\phi)$$

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Quasilinear wave equations

$$\mathbf{g}^{\alpha\beta}(\phi)\,\partial_\alpha\partial_\beta\,\phi = F(\phi, \partial\phi),$$

Energy estimates:

$$\|\phi(t)\|_s \lesssim \|\phi(0)\|_s \exp \int_0^t \|\partial\phi(\tau)\|_{L^\infty} d\tau$$

Sobolev embedding:

$$\|\partial\phi(\tau)\|_{L^\infty} \lesssim \|\phi(t)\|_s, \qquad \boxed{s > 5/2}.$$

Iteration scheme: $\phi^{(0)}, \phi^{(1)}, \ldots, \phi^{(k)},$

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Improvements based on Strichartz estimates

Strichartz: For any $s > 1$,

$$\int_0^t \|\phi(\tau)\|_{L^\infty} d\tau \lesssim \|\partial\phi(0)\|_{H^{s-1}} + \int_0^t \|\Box\phi(\tau)\|_{H^{s-1}} \qquad (1)$$

Semilinear equations: $\Box\phi = F(\phi, \partial\phi)$.
Estimate (1) can be used to improve the WP exponent to $\boxed{s > 2}$.

Quasilinear equations: $\mathbf{g}^{\alpha\beta}(\phi)\,\partial_\alpha\partial_\beta\phi = F(\phi, \partial\phi)$.
Strichartz estimates for equations with very rough coefficients (Bahouri-Chemin, Tataru, K-Rodnianski).

Theorem[K-Rodnianski, Smith-Tataru] *In wave coordinates EVE are well posed for* $\boxed{s > 2}$.

Fact: Result is sharp for general equations (Lindblad)

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Bounded L^2 curvature conjecture

Conjecture: *EVE is well posed for* $\boxed{s = 2}$, i.e. initial data sets with

$$Ric(g) \in L^2_{loc}(\Sigma_0), \quad \nabla k \in L^2_{loc}(\Sigma_0). \tag{2}$$

Fact: Need to use the special structure of the Einstein equations !

Remark: Conjecture should be viewed as a break-down criterion. *Space-time, together with a well chosen time foliation, can be extended as long as (2) hods true.*

Theorem[K-Rodnianski, Wang] *An EVE space-time, foliated by the level surfaces Σ_t of a maximal time function with future unit time normal T, can be extended as long as*

$$\int_0^t \|\mathcal{L}_T \mathbf{g}\|_{L^\infty(\Sigma_\tau)} d\tau < \infty$$

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Main Difficulties

- **Problem of coordinates** Are there coordinates, or gauge conditions, relative to which EVE exhibit some appropriate version of the null condition ?
- **Approximate solutions** Do there exist effective parametrices, for solutions of $\Box_{\mathbf{g}}\phi = 0$ with *rough metrics* $\mathbf{g}$, based on which we can prove bilinear and Strichartz estimates ?

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Cartan Formalism

Orthonormal frames: vectorfields e_α, $\alpha = 0, 1, 2, 3$,

$$\mathbf{g}(e_\alpha, e_\beta) = \mathbf{m}_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1).$$

Connection 1-forms:

$$(\mathbf{A}_\mu)_{\alpha\beta} = \mathbf{g}(\mathbf{D}_\mu e_\beta, e_\alpha)$$

Curvature:

$$\mathbf{R}_{\mu\nu\alpha\beta} = \big(\mathbf{D}_\mu \mathbf{A}_\nu - \mathbf{D}_\nu \mathbf{A}_\mu - [\mathbf{A}_\mu, \mathbf{A}_\nu]\big)_{\alpha\beta} := (\mathbf{F}_{\mu\nu})_{\alpha\beta}.$$

Yang-Mills:

$$\mathbf{D}^\mu \mathbf{F}_{\mu\nu} + [\mathbf{A}^\mu, \mathbf{F}_{\mu\nu}] = 0.$$

Frame changes: $\widetilde{e}_\alpha = \mathbf{O}_\alpha^\gamma e_\gamma$

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EVE as Yang Mills Gauge Theory

Connection 1-form: $\mathbf{A}_\mu = (A_\mu)_{\alpha\beta}$.

Curvature: $\mathbf{F}_{\mu\nu} = \mathbf{D}_\mu \mathbf{A}_\nu - \mathbf{D}_\nu \mathbf{A}_\mu - [\mathbf{A}_\mu, \mathbf{A}_\nu]$

Gauge transformations:

$$\begin{aligned}\widetilde{\mathbf{A}}_\mu &= O \cdot \mathbf{A}_\mu \cdot O^{-1} + \partial_\mu O \cdot O^{-1} \\ \widetilde{\mathbf{F}}_{\mu\nu} &= O \cdot \mathbf{F}_{\mu\nu} \cdot O^{-1}.\end{aligned}$$

Yang-Mills: $\mathbf{D}^\mu \mathbf{F}_{\mu\nu} + [\mathbf{A}^\mu, \mathbf{F}_{\mu\nu}] = 0.$

$$\begin{aligned}\square_{\mathbf{g}} \mathbf{A}_\nu - \mathbf{D}_\nu(\mathbf{D}^\mu \mathbf{A}_\mu) &= \mathbf{D}^\mu([\mathbf{A}_\mu, \mathbf{A}_\nu]) - [\mathbf{A}^\mu, \mathbf{F}_{\mu\nu}] \\ &= \mathbf{J}_\nu(\mathbf{A}, \mathbf{D}\mathbf{A})\end{aligned}$$

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Yang Mills equations in flat space.

Theorem[K-Machedon(1994)] *The flat Yang-Mills equations in* $\mathbb{R}^{1+3}$ are well posed in the energy norm, i.e $\boxed{s = 1}$.

- Use gauge freedom to impose $\sum_{i=1}^{3} \nabla^i A_i = 0$
- Equations become, with $\mathbf{A} = (A_0, A)$,

$$\begin{aligned} \Delta A_0 &= \mathbf{A} \cdot \partial \mathbf{A} + \mathbf{A}^3 \\ \Box A_i + \partial_i \partial_t A_0 &= A^j \partial_j A_i + A^j \partial_i A_j + A_0 \partial \mathbf{A} + \mathbf{A} \partial (A_0) + \mathbf{A}^3 \end{aligned}$$

- Apply $\mathcal{P} = (-\Delta)^{-1} curl\,(curl\,)$, the projection operator on the divergence free vectorfields,

$$\Box A_i = \mathcal{P}(A^j \partial_j A_i + A^j \partial_i A_j) + \text{ l.o.t.}$$

- Use bilinear estimates to control the most dangerous terms,

$$\mathcal{P}(A^j \partial_j A_i + A^j \partial_i A_j)$$

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Bilinear Estimates in $\mathbb{R}^{1+3}$

Assume: $A = (A_1, A_2, A_3)$, $\mathrm{div}\, A = 0$ in $\mathcal{D}_T = [0, T] \times \mathbb{R}^3$.

Theorem

$$\|A^i \partial_i \phi\|_{L^2(\mathcal{D}_T)} \lesssim \left(\|\partial A(0)\|_{L^2} + \int_0^T \|\Box A(t)\|_{L^2(\Sigma_t)} dt\right) \cdot \left(\|\partial \phi(0)\|_{L^2} + \int_0^T \|\Box \phi(t)\|_{L^2(\Sigma_t)} dt\right)$$

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Proof of bilinear estimates in flat space

Generalized bilinear estimates: Estimate $\|\mathcal{C}(U, \partial\phi)\|_{L^2(\mathcal{M})}$ of contractions between tensorfields U and solutions of

$$\Box\phi = 0.$$

Suffices to consider $\mathcal{C}(U, \partial\phi_f)$ with,

$$\phi_f(t,x) = \int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda u(t,x,\omega)} \widehat{f}(\lambda\omega)\lambda^2 d\lambda d\omega$$

with $u = t \pm x \cdot \omega$, i.e.

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We have,

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\begin{aligned}
\mathcal{C}(U, \partial\phi_f) &= \int_{\mathbb{S}^2} \mathcal{C}(U, \partial\, {}^{(\omega)}u) J({}^{(\omega)}u)\, d\omega \\
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Observe that ${}^{(\omega)}l := \partial\, {}^{(\omega)}u$ is the null geodesic generator of the null hyperplanes ${}^{(\omega)}u = u_0$, denoted $\mathcal{H}({}^{(\omega)}u)$.

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In interesting situations the quantity $\|\mathcal{C}(U, {}^{(\omega)}l)\|_{L^2(\mathcal{H}({}^{(\omega)}u))}$ is the flux through the null hypersurface $\mathcal{H}({}^{(\omega)}u)$ of the tensor-field U.

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Theorem (First bilinear estimate)

$$\|A \cdot \nabla\phi\|_{L^2(\mathcal{D}_T)} \lesssim \|\partial A(0)\|_{L^2(\mathbb{R}^2)} \|\partial\phi(0)\|_{L^2(\mathbb{R}^3)}.$$

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- Suffices to prove, for $\Box\phi = \Box\psi = 0, \qquad \phi = \phi_f,$

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Strategy

1. Exhibit the hidden null structure of the Einstein equations.
   - Yang-Mills formalism
   - Coulomb type gauge
   - Projection operator
2. Construct an appropriate parametrix for $\Box_{\mathbf{g}}\phi = F$. Obtain control of the error term.
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Hidden null structure of EVE.

Connection 1-form: $\mathbf{A}_\mu = (A_\mu)_{\alpha\beta} = \mathbf{g}(\mathbf{D}_\mu e_\beta, e_\alpha)$.

Curvature: $\mathbf{F}_{\mu\nu} = \mathbf{D}_\mu \mathbf{A}_\nu - \mathbf{D}_\nu \mathbf{A}_\mu - [\mathbf{A}_\mu, \mathbf{A}_\nu]$

Gauge transformations: $\widetilde{e}_\alpha = \mathbf{O}_\alpha^\gamma e_\gamma$.

$$\begin{aligned}\widetilde{\mathbf{A}}_\mu &= O \cdot \mathbf{A}_\mu \cdot O^{-1} + \partial_\mu O \cdot O^{-1}\\ \widetilde{\mathbf{F}}_{\mu\nu} &= O \cdot \mathbf{F}_{\mu\nu} \cdot O^{-1}.\end{aligned}$$

Yang-Mills: $\mathbf{D}^\mu \mathbf{F}_{\mu\nu} + [\mathbf{A}^\mu, \mathbf{F}_{\mu\nu}] = 0.$

$$\square_{\mathbf{g}} \mathbf{A}_\nu - \mathbf{D}_\nu(\mathbf{D}^\mu \mathbf{A}_\mu) = \mathbf{D}^\mu([\mathbf{A}_\mu, \mathbf{A}_\nu]) - [\mathbf{A}_\mu, \mathbf{F}_{\mu\nu}].$$

Coulomb type gauge: Choose e_0 unit normal to a maximal foliation and e_1, e_2, e_3 such that ,

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Fact: Commutation with $\mathcal{P}$ produces only null forms !

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Define

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Parametrix for $\Box_{\mathbf{g}}\phi = 0$

Define

$$\phi_f(t,x) = \int_{\mathbb{S}^2}\int_0^{\infty} e^{i\lambda u(t,x,\omega)} f(\lambda\omega)\lambda^2 d\lambda d\omega$$

$$\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$$

$$u(0,x,\omega) \sim x\cdot\omega \text{ when } |x| \to +\infty \text{ on } \Sigma_0.$$

We have,

$$\Box\phi_f = \int_{\mathbb{S}^2}\int_0^{\infty} (\Box u) e^{i\lambda u(t,x,\omega)} f(\lambda\omega)\lambda^3 d\lambda d\omega \neq 0.$$

Theorem

The operator ψ_f makes sense and,

$$\|\Box\psi_f\|_{L^2(\mathcal{M})} \lesssim \|\lambda f\|_{L^2(\mathbb{R}^3)}$$

$$\|\partial\Box\phi_f\|_{L^2(\mathcal{M})} \lesssim \|\lambda^2 f\|_{L^2(\mathbb{R}^3)}$$

Open questions

- Prove a stronger version of the global stability of Minkowski space
- Can on beat the exponent $\boxed{s=2}$?
- Find a better, **scale invariant**, continuation criterion result