

Combinatorial link Floer homology and transverse knots

Dylan Thurston

Joint with/work of Sucharit Sarkar

Ciprian Manolescu

Peter Ozsváth

Zoltán Szabó

Lenhard Ng

`math.GT/{0607691,0610559,0611841,0703446}`
<http://www.math.columbia.edu/~dpt/speaking>

June 10, 2007, Princeton, NJ

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*The invariant called knot Heegaard-Floer
Determines the genus—and more.*

*To distinguish transverse knots
(and it turns out there are lots!)*

HFK opens up a new door.

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Outline

► Introduction

Computing *HF*K

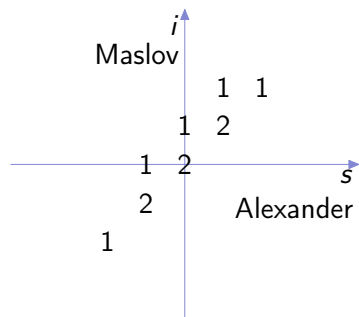
Variants

Grid moves

Transverse knots

What is Heegaard-Floer homology?

$\dim(\widehat{HFK}_i(K; s)):$



Characteristics of \widehat{HFK} :

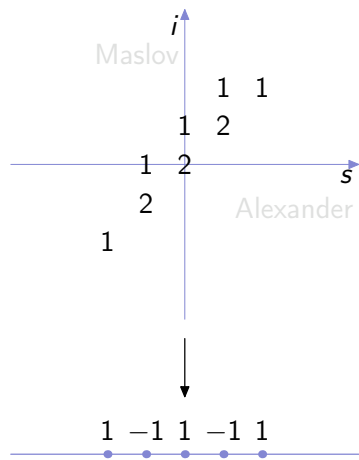
- ▶ **Bigraded;**
- ▶ Euler characteristic is Conway-Alexander polynomial;
- ▶ Max grading is knot genus; (Ozsváth-Szabó 2001)
- ▶ Determines knot fibration; (Ghiggini, Ni 2006)
- ▶ Defined via pseudo-holomorphic curves.

We will give a simple algorithm for computing HFK . . .

. . . and so the world's simplest algorithm for knot genus!

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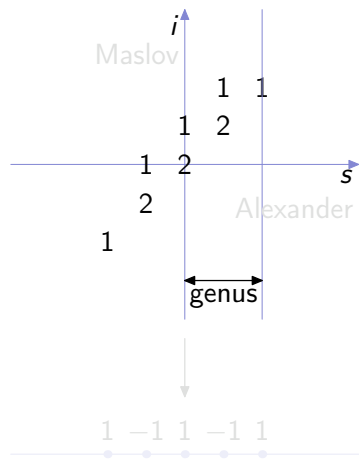
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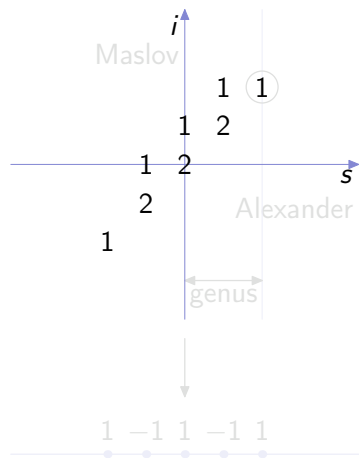
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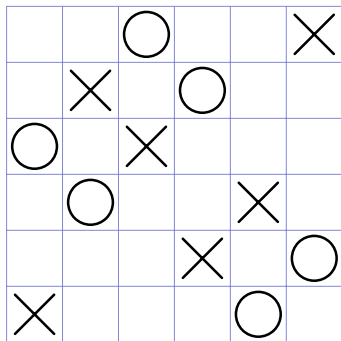
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Setting: Grid diagrams

Grid diagram: square diagram with one X and one O per row and column.



Turn it into a knot: connect
 X to O in each column;
 O to X in each row.

Cross vertical strands over horizontal.

Grid diagrams exist: take any diagram,
rotate crossings so vertical crosses over
horizontal.

The knot is unchanged under
cyclic rotations:

Move top segment to bottom.

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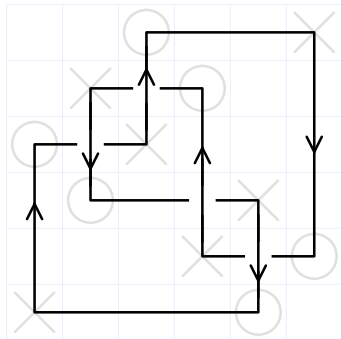
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Computing the Alexander polynomial

We categorify the following formula:

The diagram shows a 6x6 grid with a path of arrows starting from the top-left and ending at the bottom-right. The path consists of the following steps: (0,0) to (0,1) right, (0,1) to (1,1) down, (1,1) to (1,2) right, (1,2) to (2,2) down, (2,2) to (2,3) right, (2,3) to (3,3) down, (3,3) to (3,4) right, (3,4) to (4,4) down, (4,4) to (4,5) right, (4,5) to (5,5) down. The grid contains the following values:

1	1	1	t	t	t
1	1	t^{-1}	1	t	t
1	t	1	1	t	t
1	t	t	t	t^2	t
1	t	t	t	t	1
1	1	1	1	1	1

The path is indicated by arrows: (0,0) to (0,1) right, (0,1) to (1,1) down, (1,1) to (1,2) right, (1,2) to (2,2) down, (2,2) to (2,3) right, (2,3) to (3,3) down, (3,3) to (3,4) right, (3,4) to (4,4) down, (4,4) to (4,5) right, (4,5) to (5,5) down.

$$= \pm t^*(1-t)^{n-1} \Delta(K; t)$$

- ▶ Make matrix of $t^{-\text{winding \#}}$
(with extra row/column of 1's);
- ▶ det determines the Conway-Alexander polynomial Δ
($n = \text{size of diagram}$; here 6)

Outline

Introduction

▶ **Computing HFK**

Variants

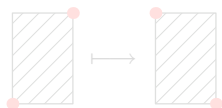
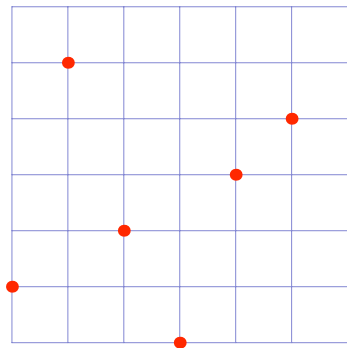
Grid moves

Transverse knots

Computing *HF*K: Chain complex \widetilde{CK}

Define a chain complex \widetilde{CK} over $\mathbb{Z}/2$.

- ▶ Generated by matchings between horizontal and vertical gridcircles (as counted in *det* for Alexander).
- ▶ Boundary ∂ switches corners on *empty rectangles*:

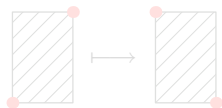
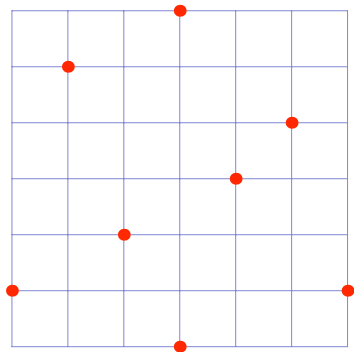


Sum over all ways to switch SW-NE corners of an empty rectangle to NW-SE corners. (*Empty* means: no *X*'s, *O*'s, or other points in generator.)

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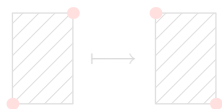
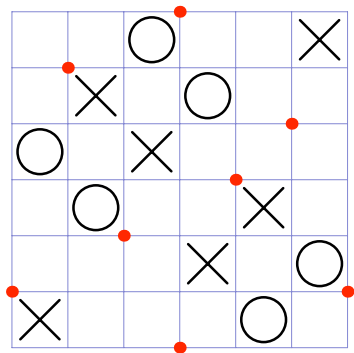


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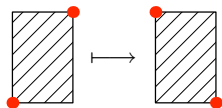
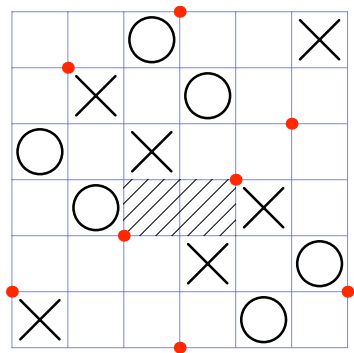


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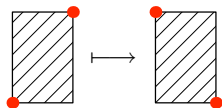
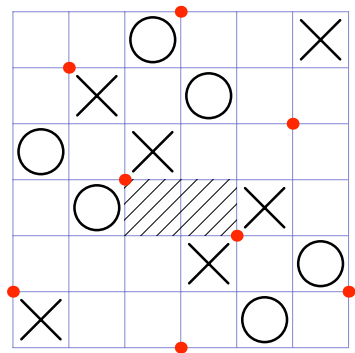


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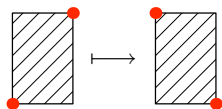
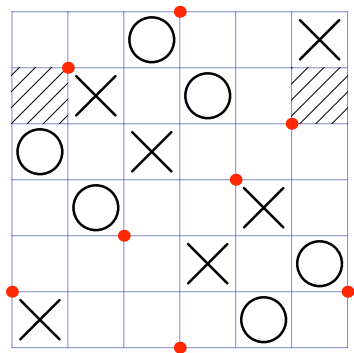


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Computing $HF\tilde{K}$: Chain complex $\tilde{C}K$

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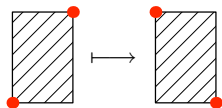
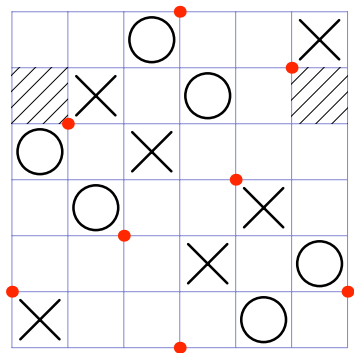


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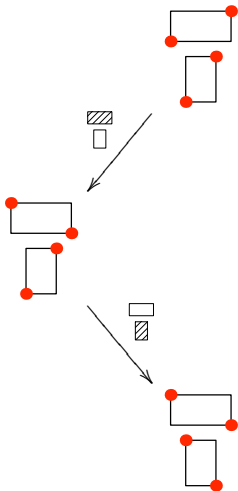
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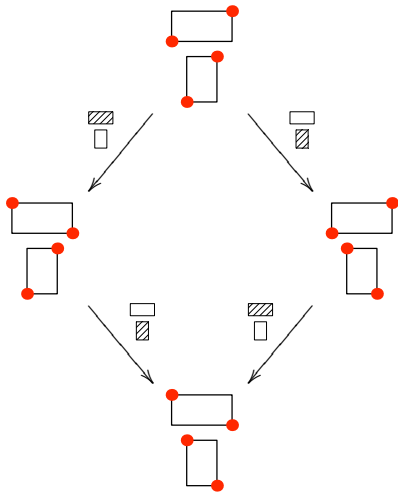
Computing HFK : $\partial^2 = 0$



Each term in ∂^2 must have a mate:

- ▶ If rectangles are disjoint, take rectangles in either order.
- ▶ If rectangles share a corner, decompose the union in another way.

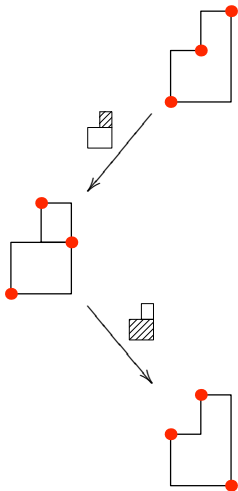
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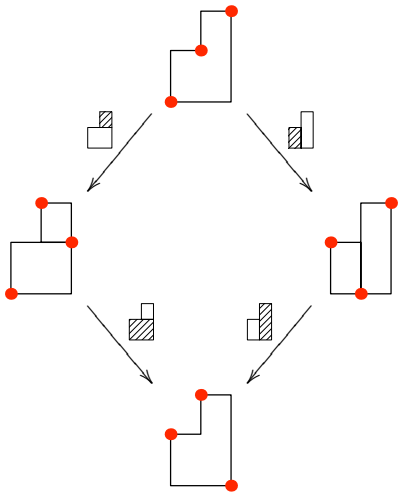
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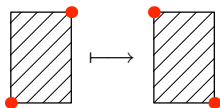


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Computing HFK : Gradings on \widetilde{CK}

In the plane,



removes one *inversion*.

For $A, B, C \subset \mathbb{R}^2$,

$$\begin{aligned} \mathcal{I}(A, B) &:= \#\{ a \square b \mid a \in A, b \in B \} \\ \mathcal{I}(A - B, C) &:= \mathcal{I}(A, C) - \mathcal{I}(B, C) \end{aligned}$$

For \mathbf{x} a generator, \mathbb{X} the set of X 's, \mathbb{O} the set of O 's, the gradings are:

► **Maslov:** $M(\mathbf{x}) := \mathcal{I}(\mathbf{x} - \mathbb{O}, \mathbf{x} - \mathbb{O}) + 1$.

► **Alexander:**

$$A(\mathbf{x}) := \frac{1}{2} (\mathcal{I}(\mathbf{x} - \mathbb{O}, \mathbf{x} - \mathbb{O}) - \mathcal{I}(\mathbf{x} - \mathbb{X}, \mathbf{x} - \mathbb{X}) - (n - 1)).$$

Computing *HF*K: The answer

Theorem (Manolescu-Ozsváth-Sarkar)

For G a grid diagram for K ,

$$H_*(\widetilde{CK}(G)) \simeq \widehat{HF}K(K) \otimes V^{\otimes n-1}$$

where $V := (\mathbb{Z}/2)_{0,0} \otimes (\mathbb{Z}/2)_{-1,-1}$.

Gillam and Baldwin used this to compute $\widehat{HF}K$ for all knots with ≤ 11 crossings, including new values of knot genus.

Outline

Introduction

Computing *HF*K

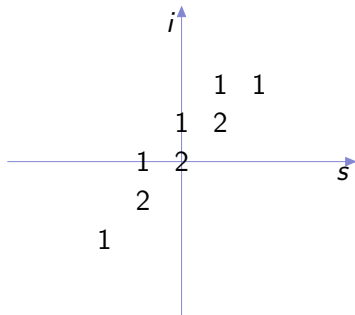
▶ **Variants**

Grid moves

Transverse knots

Improving the answer

$\dim \widehat{HFK}_i(K; s):$



To remove factors of $V^{\otimes n-1}$:

HFK^- : variant of \widehat{HFK}

Module over $\mathbb{Z}/2[U]$

U has degree $(-1, -2)$

Related to \widehat{HFK} by Univ. Coeff. Thm.

To compute: Add one U_i for each O

Complex $CK^-(G)$ over $\mathbb{Z}/2[U_1, \dots, U_n]$

∂ counts rects. that contain only O 's,
weighted by corresponding U_i .

Theorem

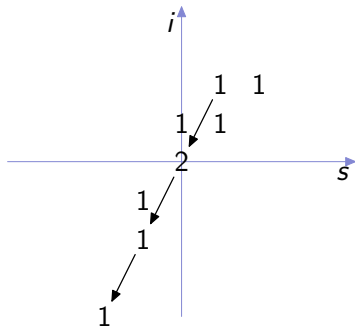
(Manolescu-Ozsváth-Sarkar)

$$H_*(CK^-(G)) \simeq HFK^-(K),$$

Each U_i acts by U on the homology.

Improving the answer

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Theorem

(Manolescu-Ozsváth-Sarkar)

$$H_*(CK^-(G)) \simeq \widehat{HFK}^-(K),$$

Each U_i acts by U on the homology.

Further variants

Can also:

- ▶ Allow rectangles to cross X 's to get a filtered complex, and
- ▶ Add signs (in essentially unique way) to work over $\mathbb{Z}[U]$.

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Computing *HF*K

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▶ **Grid moves**

Transverse knots

Combinatorial invariance

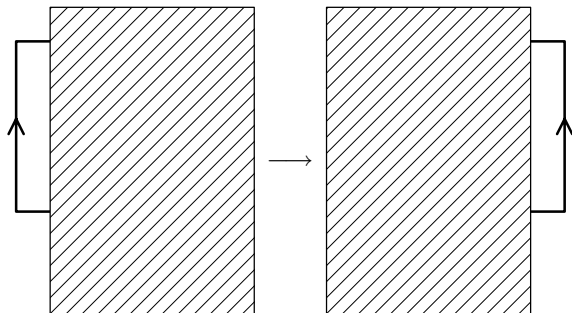
Theorem (Manolescu-Ozsváth-Szábo-T.)

For any sequence of elementary grid moves, there is an explicit chain map exhibiting invariance of HFK^- .

Conjecture (Naturality or Functoriality)

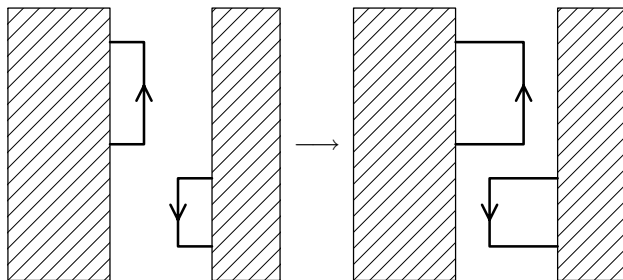
The chain map depends only on isotopy class of sequence of elementary grid moves. That is, oriented mapping class group of K acts on $HFK^-(K)$.

Elementary grid moves



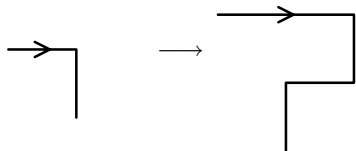
- ▶ **Cycle:** Move left column to right, or top row to bottom.
- ▶ **Commute:** Switch two non-interfering columns or rows.
- ▶ **Stabilize:** Introduce a notch at a corner.

Elementary grid moves



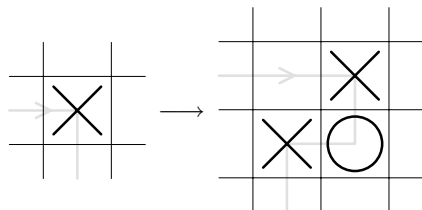
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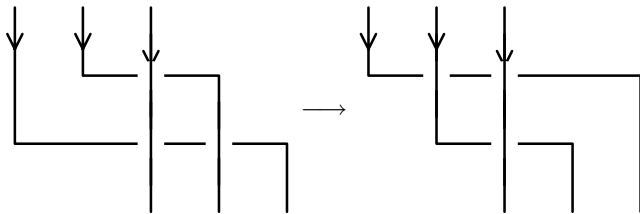
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(Cromwell '95, Dynnikov '06)

Elementary grid moves

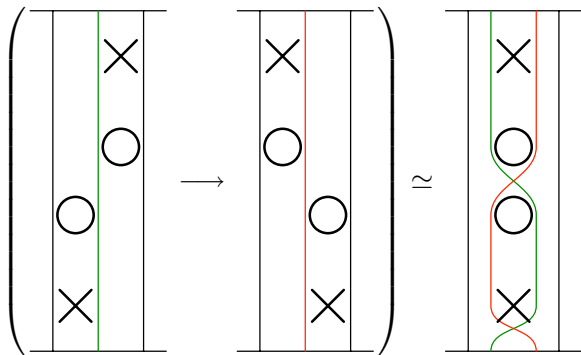


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Where's Reidemeister III?

(Cromwell '95, Dynnikov '06)

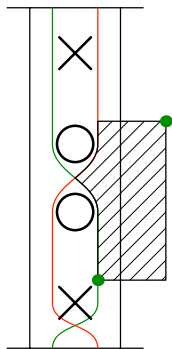
Chain map for commutation counts pentagons



To construct a chain map for commutation, draw two versions of the middle gridcircle on a single diagram.

The chain map counts empty pentagons going between the two gridcircles.

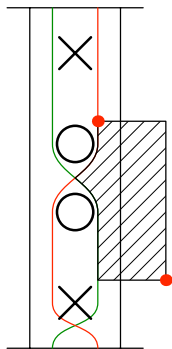
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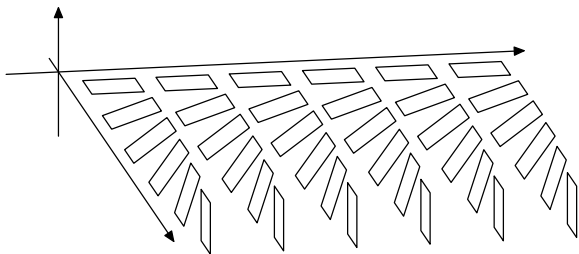
Computing *HF*K

Variants

Grid moves

▶ **Transverse knots**

Contact structures and knots



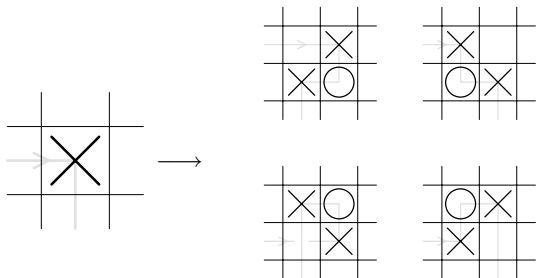
A *contact structure* is a twisted 2-plane field:
if α is a 1-form defining the plane field, $\alpha \wedge d\alpha$ is positive.
(Warning: above contact structure is reversed.)

A *Legendrian knot* is a knot that is tangent to the plane field.
A *transverse knot* is a knot that is transverse to the plane field.

Transverse knots have one easy invariant, the *self-linking number*.

Question. Can we find transverse knots with the same classical knot type and self-linking number?

Ways to stabilize

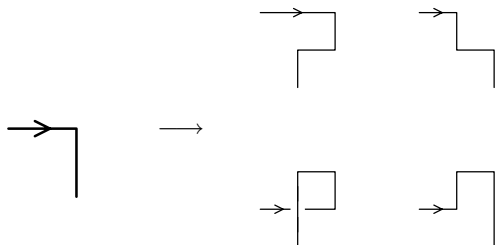


Four ways to stabilize: Where to leave the empty square?

- ▶ Two diagonal opposite ways preserve Legendrian knot.
- ▶ Two adjacent ways preserve closed braid.
- ▶ Three ways preserve transverse knot.

Warning: The Legendrian/transverse knots are mirrored.

Ways to stabilize

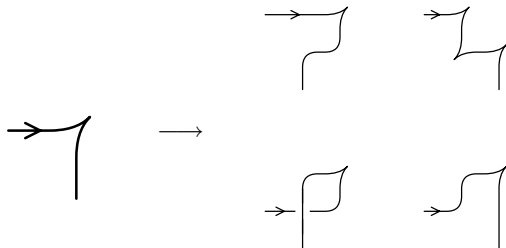


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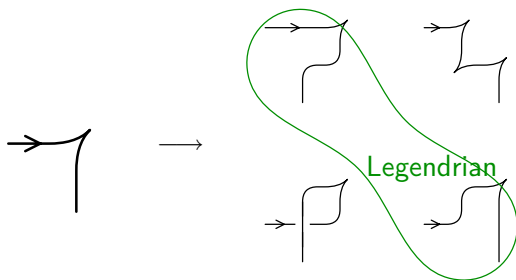


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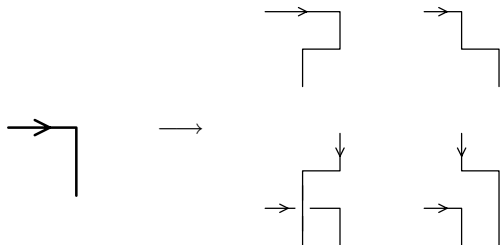


Four ways to stabilize: Where to leave the empty square?

- ▶ Two diagonal opposite ways preserve Legendrian knot.
- ▶ Two adjacent ways preserve closed braid.
- ▶ Three ways preserve transverse knot.

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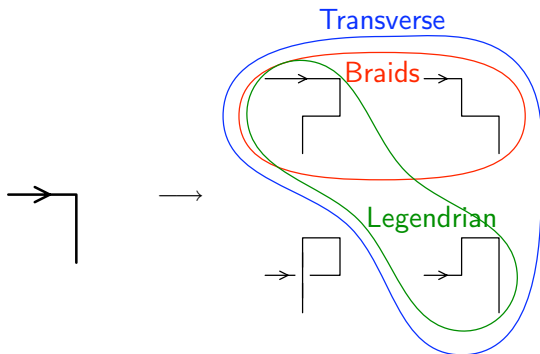


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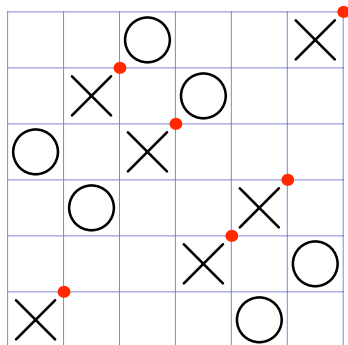


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Transverse invariant: Definition



Definition

The *canonical generator* $\mathbf{x}^+(G)$ is given by the upper-right corner of each X .

Facts:

- ▶ $\partial \mathbf{x}^+ = 0$. (The X 's block any rectangles.)
- ▶ $[\mathbf{x}^+(G)]$ maps to $[\mathbf{x}^+(G')]$ under commutation and 3 out of 4 stabilizations.

Theorem (Ozsváth-Szabó-T.)

$[\mathbf{x}^+(G)]$ in $\text{HFK}^-(m(K))$ is an invariant of the transverse knot represented by G , up to quasi-isomorphism of filtered complexes.

Transverse invariant: Properties

Let G be a grid diagram representing the transverse knot \mathcal{T} .

- ▶ $\mathbf{x}^+(G)$ lives in bigrading $(s, 2s)$, where $s = \frac{sl(\mathcal{T})+1}{2}$.
- ▶ If \mathcal{T}' differs from \mathcal{T} by a positive stabilization, then $[\mathbf{x}^+(\mathcal{T}')] = U[\mathbf{x}^+(\mathcal{T})]$.
- ▶ $[\mathbf{x}^+(\mathcal{T})] \neq 0$ in HFK^- .

Corollary

For any transverse knot \mathcal{T} of topological type K ,

$$\frac{sl(\mathcal{T}) + 1}{2} \leq \tau(K) \leq g_4(K)$$

where $\tau(K)$ is the largest Alexander grading which has an element which is not U torsion.

Transverse invariant: Examples

Let $\theta(\mathcal{T})$ (resp. $\widehat{\theta}(\mathcal{T})$) be the transverse invariant in $HFK^-(m(K))$ (resp. $\widehat{HFK}(m(K))$).

$\widehat{\theta}(\mathcal{T}) = 0$ iff $\theta(\mathcal{T})$ is divisible by U .

Theorem (Ng-Ozsváth-T.)

The knots $m(10_{132})$ and $m(12n_{200})$ have two trans. reps. with same sl , one with $\widehat{\theta} = 0$ and one with $\widehat{\theta} \neq 0$.

This technique also works for the $(2, 3)$ cable of the $(2, 3)$ torus knot, originally found by Etnyre-Honda and Menasco-Matsuda.

Let δ_1 be the next differential in the spectral sequence on \widehat{HFK} .

Theorem (Ng-Ozsváth-T.)

The pretzel knots $P(-4, -3, 3)$ and $P(-6, -3, 3)$ have two trans. reps. with same sl , one with $\delta_1 \circ \widehat{\theta} = 0$ and one with $\delta_1 \circ \widehat{\theta} \neq 0$.

Transverse invariant: Going further

Theorem (Ng-Ozsváth-T.)

If the Naturality Conjecture is true, then the twist knot 7_2 has two trans. reps. with the same sl , with $\widehat{\theta}$ in different orbits of the mapping class group.

But θ is not a complete invariant: Birman and Menasco have classified closed 3-braids up to transverse isotopy.

In their small examples of distinct transverse knots, θ lives in a 1-dimensional space, so cannot distinguish them.