

JACOBIAN DETERMINANTS AND NULL LAGRANGIANS

The Art of Integration by Parts

Tadeusz Iwaniec (*Syracuse University*)

The term null Lagrangian pertains to a nonlinear differential expression whose integral means over any subdomain can be reduced via integration by parts to the boundary. An important special case is furnished by the Jacobian determinant. In this case the familiar equality of the integral means;

$$\int_{\Omega} \mathbf{J}(x, f) \, dx = \int_{\Omega} \mathbf{J}(x, h) \, dx, \quad \text{whenever } f = h \text{ on } \partial\Omega$$

lies fairly deep in the concept of the topological degree. Many more differential expressions enjoy an identity such as this. Null Lagrangians owe much of their importance to recent advances in the calculus of variations (polyconvex energy integrals), nonlinear PDEs (compensated compactness), geometric function theory (quasiconformal deformations) and some fields of applied mathematics: nonlinear elasticity, material science and crystals (rank-one connections). Nonlinear elasticity grew out of the classical task to minimize the energy of a deformation $f : \Omega \rightarrow \mathbb{R}^n$,

$$\mathcal{E}[f] = \int_{\Omega} \mathbf{E}(x, \nabla f) \, dx,$$

subject to prescribed boundary values. Adding null Lagrangian to any such integrand does not affect the solutions, the variational Lagrange equation remains unchanged. A distinctive feature of null Lagrangians is that they possess higher degree of summability, slightly better than naturally expected. Any improvement of regularity, even the slightest possible, turns out to be a useful bonus for applications. The present lecture will certainly shed fresh light on these topics. The talk is accessible to graduate students.

Prerequisites: Multivariable Calculus, Differential Forms, Sobolev Spaces, Distributions and basics in PDEs.