Multiple Time Scale Numerical Methods for Oscillatory ODE

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ABSTRACT

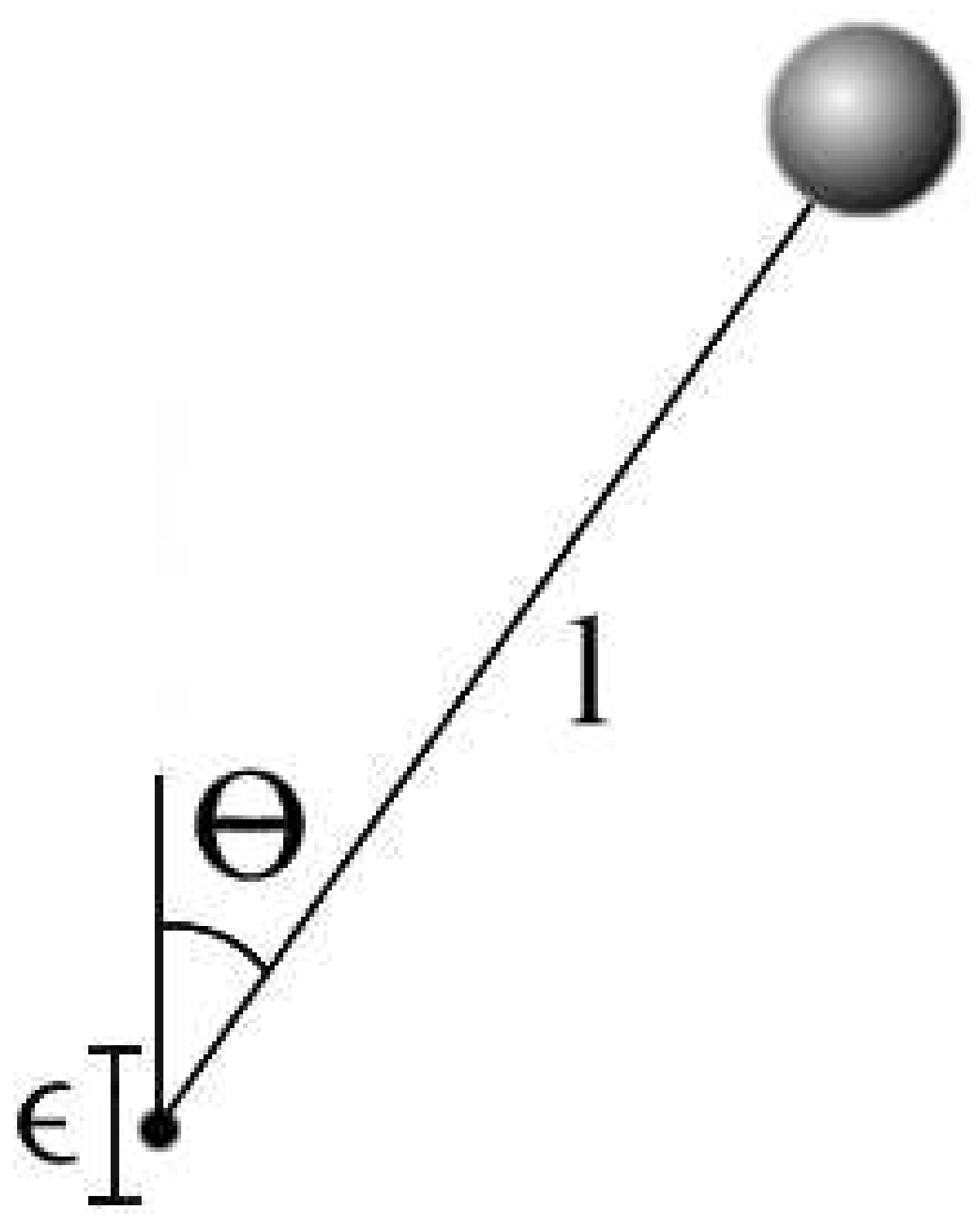
We use the heterogeneous multiscale method (HMM) framework to build numerical ODE schemes that operate on two time scales. Traditional ODE methods for stiff systems require fine time-steps and may fail over long periods. We use a smoothing kernel to calculate average forces during a short period allowing a long time step. We develop and analyze algorithms which compute the average path of an inverted pendulum under a highly oscillatory forcing of the pivot.

Introduction

- Multiscale problems of interest exhibit macroscopic motion dependent on microscopic effects.
- The microscopic scale is parametrized by $0 < \varepsilon \ll 1$.
- Traditional numerical solvers require resolution finer than the shortest wavelength (ε for the inverted pendulum) throughout the domain of interest, which is prohibitively costly.
- The Heterogeneous Multiscale Methods (HMM) tracks macroscopic variables by integrating an effective force law where any unknown data is determined by small calculations at the microscale.

Example 1: The Inverted Pendulum

- Rapid vertical forcing (frequency $O(1/\varepsilon)$) at the base of the pendulum causes stable oscillation above the horizontal axis.
- The macroscopic quantity of interest is $\Theta(t)$, the average angle between the pendulum arm and vertical.



- The equation of motion for the pendulum is,

$$l\ddot{\theta} = (g + \frac{1}{\varepsilon}\sin(2\pi\frac{t}{\varepsilon}))\sin(\theta) \quad \theta(0) = \theta_0 \quad \dot{\theta}(0) = \dot{\theta}_0. \tag{1}$$

- An analytic expression for the average motion from Levi [1],

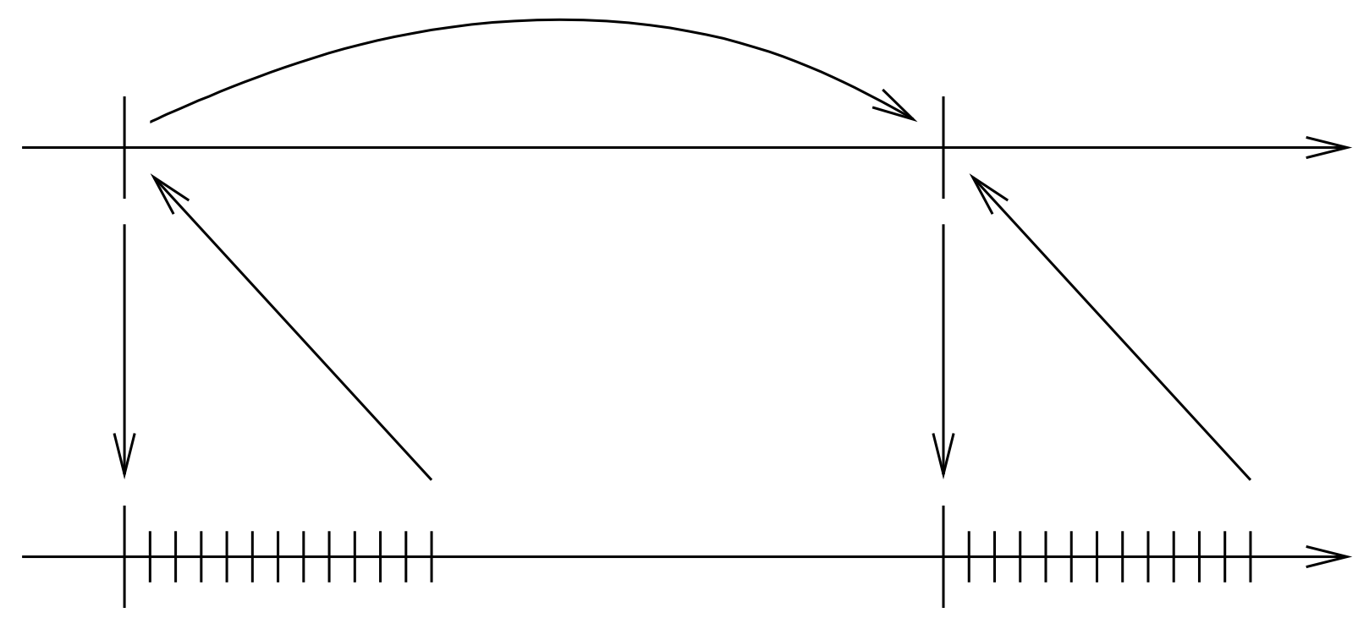
$$\ddot{\Theta} = \langle a\rangle\, f(\Theta) - \left\langle v^2\right\rangle f(\Theta) f'(\Theta) + err, \quad \Theta(0) = \Theta_0 \quad \dot{\Theta}(0) = \dot{\Theta}_0, \tag{2}$$

  where the forcing, $a(\frac{t}{\varepsilon}) = \sin(2\pi\frac{t}{\varepsilon})$, mechanical component $f(x) = \sin(x)$, and velocity

$$v(t) \quad = \quad \int_{s_0}^{t} \left(a(\frac{s}{\varepsilon}) - \langle a\rangle\right) ds. \tag{3}$$

  The choice of s_0 ensures that $\langle v\rangle = 0$, and the error term is of order $\sqrt{\varepsilon}$.

The Heterogeneous Multiscale Method



HMM is a framework for building efficient, accurate multiscale numerical methods when only a microscopic model is fully known.

- Successful applications to: Oscillatory mechanical systems, combustion models, fluid applications, ...
- Useful in multiple settings: finite difference, finite element, level-set, ...
- References [2], [3], and compare [4]

How HMM Works

- An unknown macroscopic model is assumed to exist,

$$\frac{dX}{dt} = F(X,t) \tag{4}$$

1. A standard ODE method is used to integrate Eq(4).
2. The unknown quantity $F(X,t)$ is evaluated by:
   a) Initialize a microscale calculation given the macroscopic data.
   b) Calculate the microscale solution is calculated in a small domain.
   c) Carefully average the microscale forces to extract the effective macroscopic force.
3. Use the effective force to advance the macroscopic variables.

Algorithms for the Inverted Pendulum

- Macroscopic Methods: Euler, Semi-Implicit Euler, Verlet. Semi-implicit Euler uses implicit Euler to advance the pendulum position and an explicit step to update velocity.
- Microscopic Methods: Verlet, Runge-Kutta (fourth order)
- The microscopic initial condition must be consistent with the macroscopic data, (i.e. the microscopic solution must produce the correct macroscopic average).
- Compatibility rule: Treat the macroscopic variable as constant and solve the equation $\Theta = \langle \frac{1}{\varepsilon} a(\frac{t}{\varepsilon}) f(\Theta)\rangle$ to find corrective factor.

Averaging

- The microscale dynamics have a fast oscillatory part and a slowly varying part,

$$h(t) = f(t) + g_\varepsilon(t), \quad \langle g_\varepsilon(t)\rangle = 0 \tag{5}$$

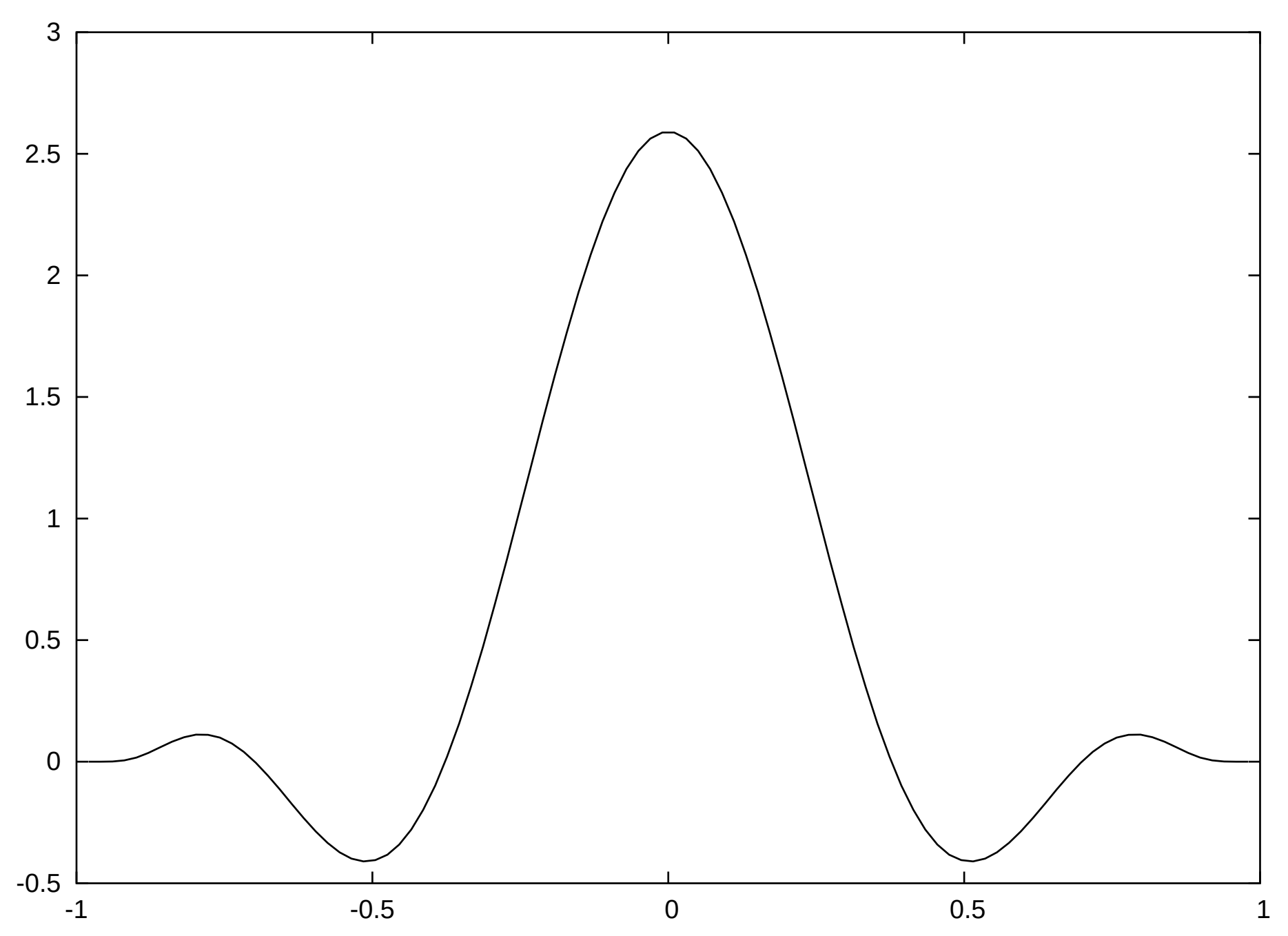
- Construct a kernel K with compact support η such that,

$$\int_\eta K(t-\tau)h(\tau)d\tau = f(t) + err. \tag{6}$$

- Error depends on the discretization of Eq.(6), and ε and η.
- Kernel performance depends on smoothness and the number of vanishing moments.
- Example form of an exponential kernel with several vanishing moments,

$$\exp\left[r\left(\left(\frac{t-t_c}{\eta}\right)^2 - 1\right)^{-1}\right]\left(\sum_{i=1}^{3} a_i \cos\left(b_i\pi\left(t-t_c\right)/\eta\right)\right) \tag{7}$$

An Example Kernel



The kernel above has the form of Eq(7). It is equal to 0 outside of the domain $[-1,1]$ and smooth at the boundary. It is normalized, and the first through fifth moments vanish.

Results for the Inverted Pendulum

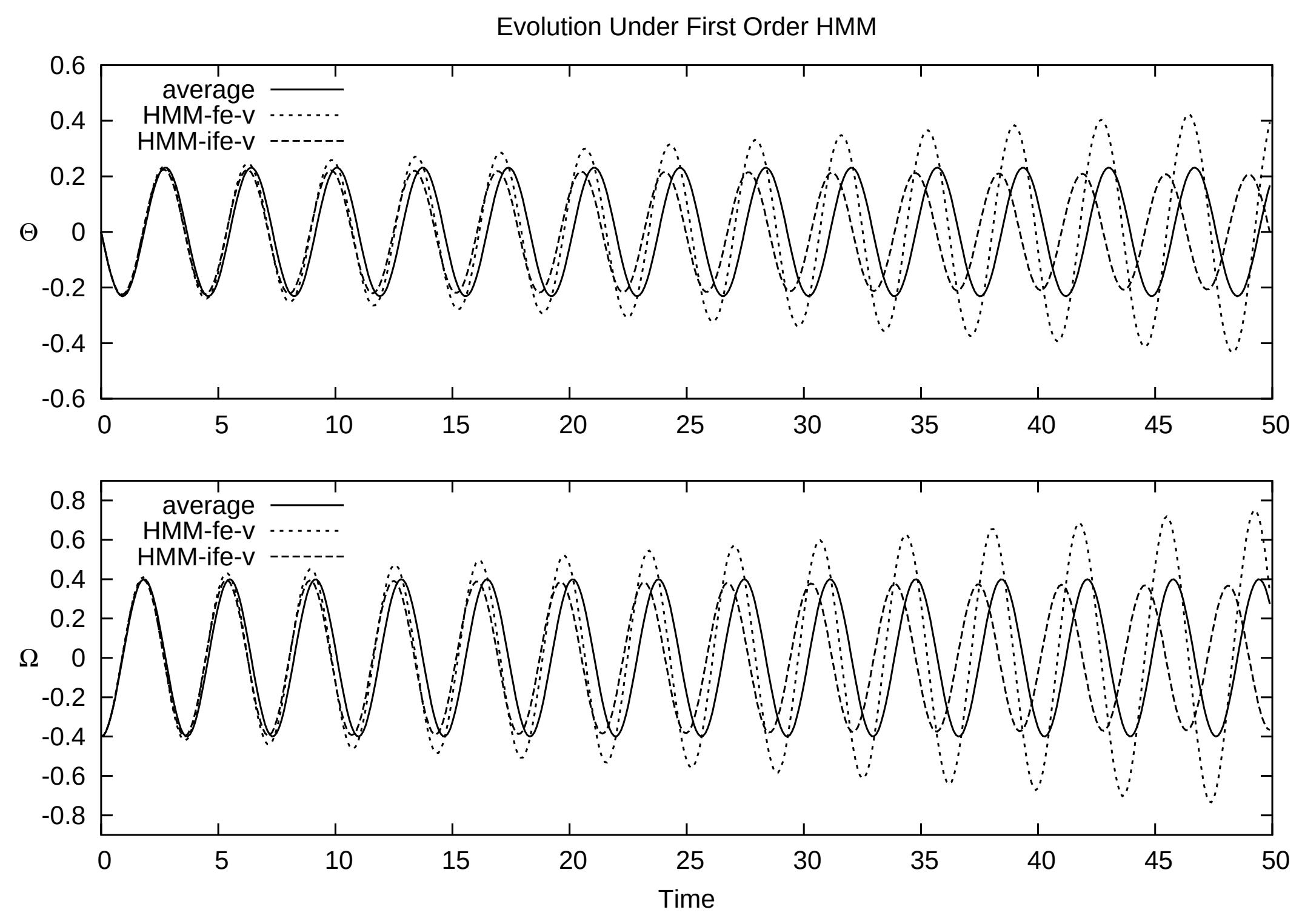
- HMM algorithms are proved to converge to Levi's analytic result [5]. That is,

$$K * \left(\left(g + \frac{1}{\varepsilon}\sin\left(2\pi\frac{t}{\varepsilon}\right)\right)\sin(\theta)\right) = \langle a\rangle f(\Theta) - \langle v^2\rangle f(\Theta) f'(\Theta) + err. \tag{8}$$

  where the error term is controlled by ε and η.

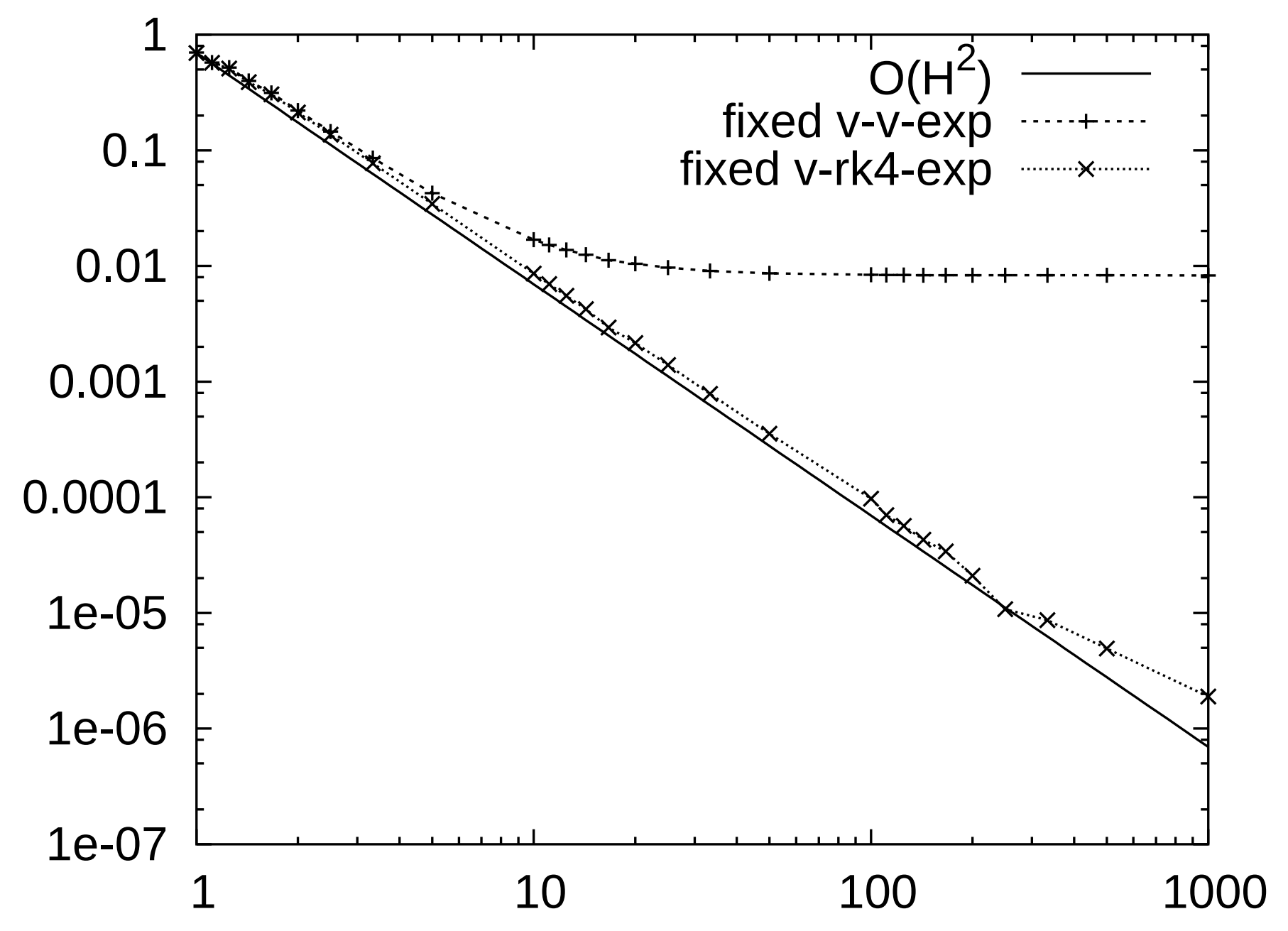
- The behavior of the error is predominantly determined by the order of the macroscopic solver.

Integration Over Long Times



The slow oscillations of the macroscopic variable as computed by Levi's solution and two first order HMM schemes ($\Omega = \dot{\Theta}$). The HMM parameters are coarse to show divergence. The HMM solutions required about 10^8 fewer evaluations of the microscale force than Euler's method would, given the same microscopic parameters.

Error Performance



Error as a function of macroscopic step size. The macroscopic solver in both cases is second order. The error levels off when a second order microscale method is used (upper curve), and maintains second order performance when the microscopic method is improved to fourth order. The abbreviations v-v-exp and v-rk4-exp name the macroscopic method, microscopic method, and kernel type in that order (v=Verlet, rk4=Runge-Kutta, exp=exponential).

Example 2: Insufficient Macroscopic Models

- Consider:

$$\frac{dX}{dt} \quad = \quad t\exp[it/\varepsilon], \quad X(0) = X_0 \tag{9}$$

$$\frac{dY}{dt} \quad = \quad |X|, \quad Y(0) = Y_0 \tag{10}$$

- Solution for X is a growing spiral.
- Assuming that the macroscopic model is a function of X and Y is insufficient. The growth of X is not captured because forces average to 0.
- Extra variables, e.g. $|X|$, could be tracked.

Summary

- HMM can be used to build efficient multiscale algorithms, which exploit scale separation in oscillatory ODE.
- The methods presented here are proved to converge to the analytic average behavior.
- Initialization of the microscopic system is non-trivial in this case.
- Future work includes extension to autonomous systems and validation of the macroscopic model.

References

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